p-Adic wavelets and their application to linear and nonlinear pseudo-differential evolutionary equations

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1. p-Adic mathematical physics. According to the well-known Ostrovsky theorem, *any nontrivial valuation on the field $\mathbb{Q}$ is equivalent either to the real valuation $|\cdot|$ or to one of the p-adic valuations $|\cdot|_p$, where p is a prime number*. This p-adic norm $|\cdot|_p$ is defined as follows: if an arbitrary rational number $x \neq 0$ is represented as $x = p^{\gamma}\frac{m}{n}$, where $\gamma = \gamma(x) \in \mathbb{Z}$ and the integers m, n are not divisible by p, then

$$|x|_p = p^{-\gamma}, \quad x \neq 0, \qquad |0|_p = 0.$$

The norm $|\cdot|_p$ satisfies the strong triangle inequality

$$|x+y|_p \leq \max(|x|_p, |y|_p).$$

The field $\mathbb{Q}_p$ of p-adic numbers is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the norm $|\cdot|_p$.

Thus there are two equal in rights universes: the real universe and the p-adic one. The latter has a specific and unusual properties. Nevertheless, there are a lot of papers where different applications of p-adic analysis to physical problems, stochastics, cognitive sciences and psychology are studied. In view of the Ostrovsky theorem, such investigations are not only of great interest in itself, but lead to applications and better understanding of similar problems in *usual* mathematical physics.

For the p-adic analysis related to the mapping $\mathbb{Q}_p \longrightarrow \mathbb{C}$, the operation of differentiation is *not defined*, and as a result, large number of models connected with p-adic differential equations use *pseudo-differential operators*. In particular, fractional operator D^{α} are extensively used in applications.

It turned out that the *wavelet analysis* is a proper technique to deal with p-adic pseudo-differential operators and equations.

It is typically that compactly supported wavelets are eigenfunctions of p-adic pseudo-differential operators, i.e., the *p-adic wavelet analysis* is the *spectral analysis* of pseudo-differential operators.

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In 2002 S. V. Kozyrev found the basis of p–adic wavelets for $L^2(\mathbb{Q}_p)$ which is an analog of the Haar basis. Some other wavelet-type systems generalizing Kozyrev's basis were suggested in the papers (J.J. Benedetto, R.L. Benedetto; Khrennikov, Kozyrev; Khrennikov, S.)

In the real analysis there is so-called *multiresolution analysis (MRA)*. It is a technique for constructing wavelet bases.

By M. Skopina and S. (2006) a concept of *p-adic multiresolution analysis (MRA)* was introduced and *all the Haar wavelet bases was described in $L^2(\mathbb{Q}_2)$*. MRA is not an identical copy of its real analog. We proved that, in contrast to Haar MRA, there exist *infinity many different Haar orthogonal bases* for $L^2(\mathbb{Q}_2)$ generated by the same MRA in $L^2(\mathbb{R})$.

2. Preliminary information and notations. The canonical form of any p-adic number $x \neq 0$ is

$$x = p^{\gamma}(x_0 + x_1 p + x_2 p^2 + \cdots),$$

where $\gamma = \gamma(x) \in \mathbb{Z}$, $x_j = 0, 1, \ldots, p-1$, $x_0 \neq 0$, $j = 0, 1, \ldots$.

The space $\mathbb{Q}_p^n = \mathbb{Q}_p \times \cdots \times \mathbb{Q}_p$ consists of points $x = (x_1, \ldots, x_n)$, where $x_j \in \mathbb{Q}_p$, $j = 1, 2 \ldots, n$, $n \geq 2$. The p-adic norm on $\mathbb{Q}_p^n$ is

$$|x|_p = \max_{1 \leq j \leq n} |x_j|_p, \quad x \in \mathbb{Q}_p^n,$$

where $|x_j|_p$ is the norm on $\mathbb{Q}_p$.

$B_{\gamma}^n(a) = \{x \in \mathbb{Q}_p^n : |x - a|_p \leq p^{\gamma}\}$ is the ball of radius p^{γ} with the center at a point $a = (a_1, \ldots, a_n) \in \mathbb{Q}_p^n$ and $S_{\gamma}^n(a) = \{x \in \mathbb{Q}_p^n : |x - a|_p = p^{\gamma}\} = B_{\gamma}^n(a) \setminus B_{\gamma-1}^n(a)$ is its boundary (sphere), $\gamma \in \mathbb{Z}$.

Let $\mathcal{E}(\mathbb{Q}_p^n)$ and $\mathcal{D}(\mathbb{Q}_p^n)$ be the linear spaces of locally-constant $\mathbb{C}$-valued functions on $\mathbb{Q}_p^n$ and locally-constant $\mathbb{C}$-valued functions with compact supports (so-called test functions), respectively. $\mathcal{D}'(\mathbb{Q}_p^n)$ is the set of all linear functionals (p-adic distributions) on $\mathcal{D}(\mathbb{Q}_p^n)$.

The Fourier transform of $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$ is defined as

$$F[\varphi](\xi) = \int_{\mathbb{Q}_p^n} \chi_p(\xi \cdot x)\varphi(x)\, d^n x, \quad \xi \in \mathbb{Q}_p^n,$$

where $d^n x = dx_1 \cdots dx_n$ is the Haar measure such that we have $\int_{|\xi|_p \le 1} d^n x = 1$; $\chi_p(\xi \cdot x) = \chi_p(\xi_1 x_1) \cdots \chi_p(\xi_n x_n)$; $\xi \cdot x$ is the scalar product of vectors and $\chi_p(\xi_j x_j) = e^{2\pi i \{\xi_j x_j\}_p}$ are additive characters of the field $\mathbb{Q}_p$, $\{x\}_p$ is a fractional part of a number $x \in \mathbb{Q}_p$. The Fourier transform is a linear isomorphism $\mathcal{D}(\mathbb{Q}_p^n)$ into $\mathcal{D}(\mathbb{Q}_p^n)$.

3. The Lizorkin spaces. To deal with pseudo-differential operators the space of distributions $\mathcal{D}'(\mathbb{Q}_p^n)$ is not good. For example, in general, $D^\alpha\varphi \notin \mathcal{D}(\mathbb{Q}_p^n)$ for $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$, i.e., the fractional operator $D^\alpha f$ is well defined only for some distributions $f \in \mathcal{D}'(\mathbb{Q}_p^n)$. To solve this problem, by Khrennikov and S. (2005) the p-adic Lizorkin spaces were introduced. The *p-adic space of Lizorkin test functions* is the following

$$\Phi(\mathbb{Q}_p^n) = \{\phi : \phi = F[\psi],\ \psi \in \Psi(\mathbb{Q}_p^n)\},$$

where $\Psi(\mathbb{Q}_p^n) = \{\psi(\xi) \in \mathcal{D}(\mathbb{Q}_p^n) : \psi(0) = 0\}$. The space $\Phi(\mathbb{Q}_p^n)$ can be equipped with the topology of the space $\mathcal{D}(\mathbb{Q}_p^n)$ which makes Φ a complete space. Since the Fourier transform is a linear isomorphism $\mathcal{D}(\mathbb{Q}_p^n)$ into $\mathcal{D}(\mathbb{Q}_p^n)$, we have

$$\phi \in \Phi(\mathbb{Q}_p^n) \iff \int_{\mathbb{Q}_p^n} \phi(x)\, d^n x = 0, \quad \phi \in \mathcal{D}(\mathbb{Q}_p^n).$$

Lemma 1. *The space $\Phi(\mathbb{Q}_p^n)$ is dense in $L^\rho(\mathbb{Q}_p^n)$, $1 < \rho < \infty$.*

The space of ***p**-adic Lizorkin distributions* $\Phi'(\mathbb{Q}_p^n)$ is the topological dual of the space $\Phi(\mathbb{Q}_p^n)$. By $\Psi^\perp$ and $\Phi^\perp$ we denote the subspaces of functionals in $\mathcal{D}'$ orthogonal to Ψ and Φ, respectively. Thus $\Psi^\perp = \{f \in \mathcal{D}'(\mathbb{Q}_p^n) : f = C\delta, C \in \mathbb{C}\}$ and $\Phi^\perp = \{f \in \mathcal{D}'(\mathbb{Q}_p^n) : f = C, C \in \mathbb{C}\}$.

Proposition 1.

$$\Phi' = \mathcal{D}'/\Phi^\perp, \qquad \Psi' = \mathcal{D}'/\Psi^\perp.$$

The space $\Phi'(\mathbb{Q}_p^n)$ can be obtained from $\mathcal{D}'(\mathbb{Q}_p^n)$ by "sifting out" constants. Thus two distributions in $\mathcal{D}'(\mathbb{Q}_p^n)$ differing by a constant are indistinguishable as elements of $\Phi'(\mathbb{Q}_p^n)$.

The *usual* Lizorkin spaces were studied in the excellent papers of P. I. Lizorkin (1963).

4. Pseudo-differential operators in the Lizorkin spaces.

By Albeverio, Khrennikov, and S. the following pseudo-differential operators in the Lizorkin space $\Phi(\mathbb{Q}_p^n)$ were introduced:

$$(A\phi)(x) = F^{-1}\big[\mathcal{A}\,F[\phi]\big](x)$$

(1)

$$= \int_{\mathbb{Q}_p^n}\int_{\mathbb{Q}_p^n} \chi_p\big((y-x)\cdot\xi\big)\mathcal{A}(\xi)\phi(y)\,d^n\xi\,d^n y, \quad \phi \in \Phi(\mathbb{Q}_p^n),$$

where $\mathcal{A} \in \mathcal{E}(\mathbb{Q}_p^n \setminus \{0\})$ is a symbol.

It is possible to define these operators on the space $\Phi'(\mathbb{Q}_p^n)$ of the Lizorkin distributions.

The space of Lizorkin distributions $\Phi'(\mathbb{Q}_p^n)$ is invariant under pseudo-differential operators A. Moreover, $A(\Phi'(\mathbb{Q}_p^n)) = \Phi'(\mathbb{Q}_p^n)$. Thus, $\Phi'(\mathbb{Q}_p^n)$ is a "natural" definition domain for pseudo-differential operators.

The family of pseudo-differential operators A with symbols $\mathcal{A}(\xi) \neq 0$, $\xi \in \mathbb{Q}_p^n \setminus \{0\}$ forms Abelian group.

Fractional operator $D^{\alpha} = F^{-1}\big[|\cdot|_p^{\alpha} F[\phi](\cdot)\big](x)$ is a particular case of pseudo-differential operator (1) with the symbol $\mathcal{A}(\xi) = |\xi|_p^{\alpha}$, $\alpha \in \mathbb{C}$.

The space of Lizorkin distributions $\Phi'(\mathbb{Q}_p^n)$ is invariant under fractional operators D^{α} and $D^{\alpha}(\Phi'(\mathbb{Q}_p^n)) = \Phi'(\mathbb{Q}_p^n)$.

The family of operators D^{α}, $\alpha \in \mathbb{C}$, forms Abelian group on the space of distributions $\Phi'(\mathbb{Q}_p^n)$.

5. Compactly supported orthonormal p-adic wavelet bases for $L^2(\mathbb{Q}_p^n)$.

Consider the set

$$I_p = \{a = p^{-\gamma}\big(a_0 + a_1 p + \cdots + a_{\gamma-1}p^{\gamma-1}\big) :$$

$$\gamma \in \mathbb{N}; a_j = 0, 1, \ldots, p-1; j = 0, 1, \ldots, \gamma - 1\}.$$

Since we have a "natural" decomposition of $\mathbb{Q}_p$ to a union of mutually disjoint discs:

$$\mathbb{Q}_p = \bigcup_{a \in I_p} B_0(a),$$

the set I_p is a "natural" set of shifts for $\mathbb{Q}_p$.

Let $\Omega(t) = 1$ for $t \in [0,\ 1]$; $\Omega(t) = 0$ for $t \notin [0,\ 1]$, $t \in \mathbb{R}$.

(a) Kozyrev (2002): the one-dimensional Haar basis

$$\theta_{k;ja}(x) = p^{-j/2}\chi_p\big(p^{-1}k(p^j x - a)\big)\Omega\big(|p^j x - a|_p\big), \tag{2}$$

where $x \in \mathbb{Q}_p$, $k = 1, 2, \ldots, p-1$, $j \in \mathbb{Z}$, $a \in I_p$.

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(b) Khrennikov, S. (2006): the multi-dimensional Haar basis

$$\Theta_{k;ja}(x) = p^{-nj/2}\chi_p\big(p^{-1}j \cdot (p^j x - a)\big)\Omega\big(|p^j x - a|_p\big), \tag{3}$$

$x = (x_1, \dots, x_n) \in \mathbb{Q}_p^n$, $\Omega(|x|_p) = \Omega(|x_1|_p) \times \cdots \times \Omega(|x_n|_p)$; $j \in \mathbb{Z}$, $a = (a_1, \dots, a_n) \in I_p^n$, $k = (k_1, \dots, k_n) \in J_{0;p}^n = \{(k_1, \dots, k_n) : j_r = 0, 1, 2, \dots, p-1; r = 1, 2, \dots, n; k_1 + \cdots + k_n \neq 0\}$, I_p^n is a tensor product of n copies of I_p.

(c) Skopina, S. (2006): all 2-adic one-dimensional Haar bases

$$\psi_{ja}^{(s)}(x) = 2^{-j/2}\psi^{(s)}(2^j x - a),$$

where

$$\psi^{(s)}(x) = \sum_{k=0}^{2^s-1} \alpha_k \psi^{(0)}\Big(x - \frac{k}{2^s}\Big), \quad s = 1, 2, \dots, \tag{4}$$

$\alpha_k = 2^{-s}(-1)^k \sum_{r=0}^{2^s-1} \gamma_r e^{-i\pi\frac{2r+1}{2^s}k}$, $k = 0, \dots, 2^s - 1$, $\gamma_r \in \mathbb{C}$, $|\gamma_r| = 1$.

(d) Skopina, S. (2006): We constructed all 2-adic multidimensional the Haar wavelet bases by means of a tensor product of one-dimensional MRAs. This standard approach for construction of multivariate wavelets was suggested by Y. Meyer.

(e) Khrennikov, S. (2006): non-Haar one-dimensional basis

$$\theta^{(m)}_{s;ja}(x) = p^{-j/2}\chi_p\big(s(p^j x - a)\big)\Omega\big(|p^j x - a|_p\big), \quad x \in \mathbb{Q}_p, \tag{5}$$

where $j \in \mathbb{Z}$, $a \in I_p$; $m \geq 1$ is a *fixed* positive integer;

$$s \in J_{p;m} = \{s = p^{-m}\big(s_0 + s_1 p + \cdots + s_{m-1}p^{m-1}\big) :$$
$$s_j = 0, 1, \ldots, p-1;\ j = 0, 1, \ldots, m-1;\ s_0 \neq 0\}.$$

(f) Khrennikov, S. (2006): non-Haar multi-dimensional wavelet basis we introduce by the n-direct product of the one- dimensional p-adic wavelets (5). For $x = (x_1, \dots, x_n) \in \mathbb{Q}_p^n$ and $j = (j_1, \dots, j_n) \in \mathbb{Z}^n$ we denote dilatation

$$\widehat{p^j} x \stackrel{def}{=} (p^{j_1} x_1, \dots, p^{j_n} x_n).$$

The n-direct product of the one-dimensional p-adic wavelets (5) are defined as

(6)

$$\Theta_{s;ja}^{(m)\times}(x) = p^{-n|j|/2} \chi_p\big(s \cdot (\widehat{p^j} x - a)\big) \Omega\big(|\widehat{p^j} x - a|_p\big), \quad x \in \mathbb{Q}_p^n,$$

where $j \in \mathbb{Z}^n$; $|j| = j_1 + \cdots + j_n$; $a = (a_1, \dots, a_n) \in I_p^n$; $s = (s_1, \dots, s_n) \in J_{p;m}^n$; $m = (m_1, \dots, m_n)$, $m_r \geq 1$ is a *fixed* positive integer, $r = 1, 2, \dots, n$. Here I_p^n, $J_{p;m}^n$ are the n-direct products of the corresponding sets.

All the above wavelet bases belong to the Lyzorkin space of test function.

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6. Wavelet analysis as a spectral analysis. *Under the proper condition, the above wavelet functions are eigenfunctions of p-adic pseudo-differential operator (1).*

Theorem 1. (Khrennikov, S., 2006) *If A is a pseudo-differential operator with a symbol $\mathcal{A}(\xi) \in \mathcal{E}(\mathbb{Q}_p^n \setminus \{0\})$, then the Haar wavelet function* (3)

$$\Theta_{k;ja}(x) = p^{-nj/2}\chi_p\big(p^{-1}k\cdot(p^j x - a)\big)\Omega\big(|p^j x - a|_p\big), \; x \in \mathbb{Q}_p^n,$$

is an eigenfunction of A if and only if

$$\mathcal{A}\big(p^j(-p^{-1}k + \eta)\big) = \mathcal{A}\big(-p^{j-1}k\big), \quad \forall\, \eta \in \mathbb{Z}_p^n; \tag{7}$$

holds, where $j \in \mathbb{Z}$, $a \in I_p^n$, $k = (k_1, \dots, k_n) \in J_{0;p}^n$. We have

$$A\Theta_{k;ja}(x) = \mathcal{A}(-p^{j-1}k)\Theta_{k;ja}(x).$$

Corollary 1. *For fractional operator D^β (the symbol $\mathcal{A}(\xi) = |\xi|_p^\beta$) we have*

$$D^\beta\Theta_{k;ja}(x) = p^{\beta(1-j)}\Theta_{k;ja}(x), \; \beta \in \mathbb{C}, \; x \in \mathbb{Q}_p^n.$$

Theorem 2. (Khrennikov, S., 2006) *If $\boldsymbol{A}$ is a pseudo-differential operator with a symbol $\boldsymbol{\mathcal{A}(\xi)} \in \boldsymbol{\mathcal{E}}(\mathbb{Q}_p^n \setminus \{0\})$, then the $\boldsymbol{n}$-dimensional non-Haaar wavelet function* (6)

$$\Theta_{s;ja}^{(m)\times}(x) = p^{-n|j|/2}\chi_p\big(s \cdot (\widehat{p^j}x - a)\big)\Omega\big(|\widehat{p^j}x - a|_p\big), \; x \in \mathbb{Q}_p^n,$$

is an eigenfunction of $\boldsymbol{A}$ if and only if

$$\mathcal{A}\big(\widehat{p^j}(-s+\eta)\big) = \mathcal{A}\big(-\widehat{p^j}s\big), \qquad \forall\, \eta \in \mathbb{Z}_p^n, \tag{8}$$

holds, where $j \in \mathbb{Z}^n$; $a \in I_p^n$; $s \in J_{p;m}^n$; $m = (m_1, \ldots, m_n)$, $m_r \geq 1$ is a fixed positive integer, $r = 1, 2, \ldots, n$. We have

$$A\Theta_{s;ja}^{(m)}(x) = \mathcal{A}(-\widehat{p^j}s)\Theta_{s;ja}^{(m)}(x).$$

Corollary 2. *The $\boldsymbol{n}$-dimensional non-Haaar wavelet function* (6) *is an eigenfunction of $\boldsymbol{D}^{\beta}$:*

$$D_x^{\beta}\Theta_{s;ja}^{(m)\times}(x) = p^{\beta \max_{1\leq k\leq n}\{m_k - j_k\}}\Theta_{s;ja}^{(m)\times}(x), \; x \in \mathbb{Q}_p^n,$$

$\beta \in \mathbb{C}$, $j \in \mathbb{Z}^n$, $a \in I_p^n$, $s \in J_{p;m}^n$.

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Theorem 3. (S., Skopina, 2006) *The* $\mathbf{2}$*-adic multidimensional Haar wavelet function is an eigenfunction of* $\boldsymbol{A}$ *if and only if a special condition satisfies.*

This the Haar wavelet function is an eigenfunction of the fractional operator.

7. Pseudo-differential evolutionary equations. Using the wavelet bases one can construct solutions of the Cauchy problems for the following pseudo-differential equations.

(a) Linear equations:

$$\frac{\partial u(x,t)}{\partial t} + A_x u(x,t) = 0,\ x \in \mathbb{Q}_p^n, \quad t \in \mathbb{R}, \quad t \geq 0,$$

and

$$i\frac{\partial u(x,t)}{\partial t} - A_x u(x,t) = 0,\ x \in \mathbb{Q}_p^n,\ t \in \mathbb{R},\ t \geq 0,$$

where $A_x u(x,t) = F^{-1}\big[\mathcal{A}(\xi)\, F[u(\cdot,t)](\xi)\big](x)$ is a pseudo- differential operator (with respect to x).

(b) Semi-linear equation:

$$\frac{\partial u(x,t)}{\partial t} + A_x u(x,t) + u(x,t)|u(x,t)|^{2m} = 0,\ x \in \mathbb{Q}_p^n,\ t \geq 0,$$

where $m \in \mathbb{N}$.

(c) Nonlinear equations:

$$\frac{\partial u(x,t)}{\partial t} + D_x^{\beta} u^2(x,t) = 0, \quad x \in \mathbb{Q}_p^n,\ t \in \mathbb{R},\ t \geq 0;$$

$$\frac{\partial u(x,t)}{\partial t} + A_x \mathcal{P}\big(u(x,t)\big) + \mathcal{R}\big(u(x,t)\big) = 0,\ x \in \mathbb{Q}_p^n,\ t \geq 0,$$

where $\mathcal{P}(z)$, $\mathcal{R}(z)$ are polynomials.

• Taking into account that the wavelet functions are eigenfunction of our pseudo-differential operators (under some conditions), we are going to seek a solution of the Cauchy problem in the form

$$u(x,t) = \sum_{k\,j\,a} \Lambda_{kja}(t) \Theta_{k;ja}(x)$$

in the Lizorkin space of distributions $\Phi'(\mathbb{Q}_p^n)$, where $\Theta_{k;ja}(x)$ is a wavelet function, $\Lambda_{kja}(t)$ are the desired functions, $j \in \mathbb{Z}$, $a \in I_p^n$, $k = (k_1, \ldots, k_n) \in J_{0;p}^n$.

• Substituting this relation into pseudo-differential equation, we obtain the system of equations for coefficients $\Lambda_{kja}(t)$. Solving this system one can construct the Cauchy problem. This idea is applicable for linear and semi-linear equations.

• To solve nonlinear problem, in addition to this approach, we must use the fact that the product of the Haar wavelet functions is the Haar wavelet function or the characteristic function of the ball.

Example Let $u^0(x) = \sum_{\gamma\in\mathbb{Z}, j\in J^n_{0;p}, a\in I^n_p} c_{\gamma j a}\Theta_{j;\gamma a}(x)$, where $c_{\gamma j a}$ are constants and $a' - p^{\gamma'-\gamma}a \notin \mathbb{Z}^n_p$ if $\gamma < \gamma'$. Then the Cauchy problem for semi-linear equation (b) with the initial data $u^0(x)$ has the unique solution (in the sense of the Lizorkin space $\Phi'(\mathbb{Q}^n_p)$)

$$u(x,t) =$$

$$\sum_{\substack{j\in J^n_{0;p},\\ \gamma\in\mathbb{Z}, a\in I^n_p}} \frac{c_{\gamma j a}\big(\mathcal{A}(-p^{\gamma-1}j)\big)^{1/2m} e^{-\mathcal{A}(-p^{\gamma-1}j)t}\Theta_{j;\gamma a}(x)}{\Big(\mathcal{A}(-p^{\gamma-1}j) + c^{2m}_{\gamma j a}p^{-mn\gamma}\big(1 - e^{-2m\mathcal{A}(-p^{\gamma-1}j)t}\big)\Big)^{1/2m}},$$

for $t \geq 0$.

Note, if at the initial instant of time $t = 0$ the initial data $u^0(x)$ is localized in some ball, solution $u(x,t)$ of the Cauchy problem conserve this localization for all $t > 0$. This fact was first observed by A. Khrennikov and S. Kozyrev.

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