Stochastic processes in Q_p associated with nonlinear PDEs

Ya. Belopolskaya (St.Petersburg) (joint with S. Albeverio) (Bonn)

1 Stochastic processes and pseudo-differential equations

We start with remaining some results about stochastic processes in the field Q_p of p-adic numbers. The details can be find in the book by Kochubej [1].

Consider the Cauchy problem

$$\frac{\partial u}{\partial s} + \frac{p^\alpha - 1}{1 - p^{-\alpha-1}} \int_{Q_p} [\varphi(x + f(s, x, u(s, x), z)) -$$

$$\varphi(x)] \|z\|_p^{-\alpha-1} dz = 0, \quad u(T, x) = \phi(x). \qquad (1.1)$$

Let Q_p denote the field of p-adic numbers – the completion of rational numbers with respect to the absolute value $\|x\|_p$ defined by setting $\|0\|_p = 0$,

$$\|x\|_p = p^{-k} \quad \mathrm{i}f \quad x = p^k \frac{m}{n}$$

where $k, m, n \in Z$ and m, n are prime to p.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $\nu_\alpha(t, \Gamma)$ be random measure non-negative countably additive on the Borel σ-algebra $\mathcal{B}(Q_p\backslash 0)$ and $\nu_\alpha(t,\Gamma_1), \nu_\alpha(t,\Gamma_2)$ be independent if $\Gamma_1 \cap \Gamma_2 = \emptyset$. Below we assume that

$$E\nu_\alpha(t,\Gamma) = t\pi_\alpha(\Gamma), \quad \pi_\alpha(dz) = \frac{p^\alpha - 1}{1 - p^{-\alpha-1}} \|z\|_p^{-\alpha-1} dz.$$

where π_α is called the Lévy measure.

Denote by $\mathcal{M}(\mathcal{F}_t)$ the set of all predictable step functions such that $f(t,z) = 0$ for $z \in B_k$ where the integer k depends on f, $B_k = \{z \in Q_p : \|z\|_p \leq p^k\}$.

Let $C(S, Q_p)$ denote the space of all continuous Q_p-valued functions defined on a compact set $S \subset Q_p$ and $D([0,T], X)$ denote the space of cadlag X-valued functions defined on $[0,T]$. Note that functions from $D([0,T], Q_p)$ are constant between their jumps because of total disconnectedness of Q_p.

Let $f(t,z) = \Sigma_{k=1}^{\infty} a_k(t) z^k, \quad z \in Q_p, t \in [0,T]$ where $a_k(t) \in D([0,T], Q_p)$ and $\lim_{k\to\infty} sup_0 \|a_k(t)\|_p^{\frac{1}{k}} = 0$. Then $f \in D([0,T], C(S_n, Q_p))$ for each integer n, where $S_n = \{z \in Q_p : \|z\|_p = p^n\}$.

Denote by $I(f)$ a stochastic integral of the form

$$I(f) = \int_0^T \int_{Q_p} f(t,z)\nu_\alpha(dt,dz) \tag{1.2}$$

and recall that it is defined first on S_n as

$$I_n(f_m) = \int_0^T \int_{S_n} f_m(t,z)\nu_\alpha(dt,dz) =$$

$$\sum_{i=1}^{N_m} \int_{S_n} f_{m,i}(z)\nu_\alpha(\Delta_i^{(m)}, dz)$$

It results from the properties of the integral over S_n

$$\|I(f)\|_p \leq C \sup_{x\in S_n} \|f(x)\|_p$$

and the sequence $I_{n,m} = I_n(f_m)$ is fundamental in Q_p almost surely. Consider $I_n(f) = \lim_{m\to\infty} I_{n,m}$ and $I(f) = \Sigma_{n=-\infty}^{\infty} I_n(f)$ where the series converges in probability.

To define a stochastic integral $I(f)$ for a random function $f \in Q_p$ we assume that $f \in \mathcal{M}_t$ – the space of $\mathcal{F}_t$- measurable step functions such that $f(t,z) = 0$ for $z \in S_k$ where k depends on f. For any $f \in \mathcal{M}_t$ the stochastic integral is defined as

$$I(f) = \int_0^T \int_{Q_p} f(t,z)\nu_\alpha(dt,dz) =$$

$$\sum_{k=0}^{n} \sum_{j=1}^{m} f(t_k, z_j)\nu_\alpha([t_k, t_{k+1}], G_j)$$

where G_j are Borel subsets of Q_p , $\cup_{j=1}^{m} G_j = Q_p$.

For any $f \in \mathcal{M}_t$ the following estimate holds

$$E\|I(f)\|_p \leq C \int_0^T \int_{Q_p} \|f(t,z)\|_p \pi_\alpha(dz)dt. \qquad (1.3)$$

If the process ξ has a finite variation with jump times T_i and X_i denote the jump sizes then

$$\xi(t) = \sum_{i=1}^{N_t} X_i \to [\xi,\xi]_t = \sum_{i=1}^{N_t} \|X_i\|_p^2 = \sum_{0<s<t} \|\Delta\xi(s)\|_p^2.$$

Below we assume that the following assumptions hold.

C 1.1. $f(\cdot,x,z) \in D([0,T], C_b(Q_p \times S_n, L_\gamma(\Omega, Q_p)))$ for every $n \in Z$ where C_b is the space of bounded continuous vector functions with sup norm.

C 1.2. The mapping $z \to f(t,x,z)$ from Q_p to $L_\gamma(\Omega, Q_p)$ is continuous uniformly with respect to $t \in [0,T]$, $x \in Q_p$.

C 1.3. $\int_{Q_p} \sup_{x\in Q_p, t\in[0,T]} \|f(t,x,z)\|_p^\gamma \|z\|_p^{-\alpha-1} dz < \infty.$

C 1.4. For each $n \in Z$, there exists a constant $L_n > 0$ such that for all $t \in [0,T]$, $\quad x, y \in Q_p, z \in S_n$ such that

$$\|f(t,x,z) - f(t,y,z)\|_p \leq L_n \|x-y\|_p \qquad (1.4)$$

and

$$\sum_{n=-\infty}^{\infty} L_n p^{-n(\alpha+1)} \leq C < \infty.$$

We denote by $\mathcal{X}$ the Banach space of (classes of stochastically equivalent) Q_p-valued processes $\xi(t)$, $t \in [0,T]$ $\mathcal{F}_t$ measurable for each t continuous as functions on $[0,T]$ with values in $L^1(\Omega, Q_p)$ equipped with the norm

$$\|\xi\|_{\mathcal{X}} = \sup_{0 \leq t \leq T} E\|\xi(t)\|_p.$$

Consider the basic jump process $\zeta(t)$ and denote by θ_k jump times of this process such that

$$\|\zeta(\theta_k) - \zeta(\theta_k - 0)\| > p^{-n}.$$

Given a step function f_n consider a stochastic process

$$\xi(t) = x + \int_0^t \int_{Q_p} f_n(\theta, z) \nu_\alpha(dt, dz)$$

and note that the process $\xi(t)$ can be written in the form

$$\xi(t) = x + \sum_{\theta_m \leq t} f_n(\theta_m, z_m)$$

where $z_m = \zeta(\theta_m) - \zeta(\theta_m - 0)$. The process $\xi(t)$ has almost surely a finite number of jumps on the interval $[0,t]$ that allows to present the increment of $\varphi(\xi(t))$ as a sum of corresponding jumps. Thus we get

$$\varphi(\xi(t)) - \varphi(x) =$$

$$\int_0^t \int_{Q_p} [\varphi(\xi(\theta - 0) + f(\theta, z)) - \varphi(\xi(\theta - 0))] \nu_\alpha(d\theta, dz).$$

This formula is called the Ito formula for the jump process $\xi(t)$.

Consider a stochastic equation

$$\xi(t) = x + \int_s^t \int_{Q_p} f(\theta, \xi(\theta), z)\nu_\alpha(d\theta, dz) \qquad (1.5)$$

Theorem 1.1. *Under the assumptions* **C 1.1** – **C 1.4** *there exists a unique solution* $\xi \in \mathcal{X}$ *of (1.5).*

As soon as $f(t, x, z)$ is nonrandom it is known that the solution $\xi(t)$ of (1.5) possesses the Markov property.

Denote by $H(Q_p)$ the set of real valued bounded functions on $[0, T] \times Q_p$ satisfying the Lipschitz condition

$$|v(t, x) - v(t, y)| \leq C_n^v(t)\|x - y\|_p$$

if $x, y \in S_n$, $t \in [0, T]$ and

$$\sum_{n=-\infty}^{\infty} C_n^v(t) p^{-n(\alpha+1)} < \infty.$$

One can verify that the relation

$$u(s, x) = E[\varphi(\xi_{s,x}(t))] \qquad (1.6)$$

gives rise to an evolution family in $H(Q_p)$. If $\varphi \in H(Q_p)$ then

$$\mathcal{A}\varphi(x) = \lim_{t \to s} \frac{E\varphi(\xi_{s,x}(t)) - \varphi(x)}{t - s} =$$

$$\frac{p^\alpha - 1}{1 - p^{-\alpha-1}} \int_{Q_p} [\varphi(x + f(s, x, z)) - \varphi(x)] \|z\|_p^{-\alpha-1} dz \qquad (1.7)$$

uniformly with respect to $x \in Q_p$. The operator $\mathcal{A}$ defined by (1.7) is called the generator of the Markov process $\xi_{s,x}(t)$.

Finally the following statement holds.

Theorem 1.2. *Under the conditions of theorem 1.1 the function*

$$u(s, x) = E[\varphi(\xi_{s,x}(T))] \qquad (1.8)$$

determines a unique solution to the Cauchy problem

$$\frac{\partial u(s,x)}{\partial s} + \mathcal{A}u(s,x) = 0, \quad u(T,x) = \varphi(x), \tag{1.9}$$

where $\mathcal{A}$ is defined by (1.7).

One can consider a more general Cauchy problem of the form

$$\frac{\partial u(s,x)}{\partial s} + \mathcal{A}u(s,x) + c(x)u(s,x) = 0, \quad u(T,x) = \varphi(x), \tag{1.10}$$

where $c(x)$ is a bounded scalar function defined on Q_p. The famous Feynman-Kac formula is valid in this case as well. Namely the following statement holds.

Theorem 1.3. *Let $c(x)$ be a bounded scalar function defined on Q_p. Then under the conditions of theorem 1.1 the function*

$$u(s,x) = E[e^{\int_s^T c(\xi(\tau))d\tau}\varphi(\xi_{s,x}(T))] \tag{1.11}$$

determines a unique solution to the Cauchy problem (1.10).

2 Stochastic processes associated with nonlinear evolution families

Consider a nonlinear parabolic equation

$$\frac{\partial u}{\partial t} + (u, \nabla)u = \frac{\sigma^2}{2}\Delta u,$$

called the Burgers equation and construct a solution of the Cauchy problem for this equation with the Cauchy data $u(0,x) = u_0(x)$, $x \in R^d$. Next consider a stochastic system

$$d\xi(\tau) = -u(t-\tau, \xi(\tau))d\tau + \sigma dw(\tau), \quad \xi(0) = x$$

$$u(t,x)=E[u_0(\xi(t))].$$

The solution of this stochastic system determines the solution of the Cauchy problem for the Burgers equation [2].

Our purpose now is to construct a stochastic process associated with the Cauchy problem solution

$$\frac{\partial u(s,x)}{\partial s}+\mathcal{A}^u u(s,x)=0,\quad u(T,x)=\varphi(x) \qquad (2.1)$$

where

$$\mathcal{A}^u v(s,x)=$$

$$\frac{p^\alpha-1}{1-p^{-\alpha-1}}\int_{Q_p}[v(s,x+f(x,u(s,x),z))-v(s,x)]\|z\|_p^{-\alpha-1}dz. \qquad (2.2)$$

Note that in the case when R^d-valued processes are considered instead of Q_p valued ones the similar problem was solved in [3],[4]. To construct such a process in Q_p we consider the system

$$\xi(t)=x+\int_s^t\int_{Q_p}f(\xi(\theta),u(\theta,\xi(\theta)),z)\nu_\alpha(d\theta,dz), \quad (2.3)$$

$$u(s,x)=E[\varphi(\xi_{s,x}(T))], \qquad (2.4)$$

prove the existence and uniqueness of its solution and then check that $u(s,x)$ given by (2.4) satisfies the Cauchy problem (2.1).

To realize this program we consider the successive approximations

$$u^1(s,x)=\varphi(x), \qquad (2.5)$$

$$\xi^n(t)=x+\int_s^t\int_{Q_p}f(\xi^n(\tau),u^n(\tau,\xi^n(\tau)),z)\nu_\alpha(d\tau,dz), \qquad (2.6)$$

$$u^{n+1}(s,x) = E\varphi(\xi^n(t)) \tag{2.7}$$

and prove that the sequence $(\xi^n(t), u^n(s,x))$ converges to a limit $(\xi(t), u(s,x))$.

We say that condition **C 2.1** holds if

$$\|f(x,v,z) - f(x_1,v_1,z)\|_p \le C_f[1 + L_n\|x-x_1\|_p + M_n|v-v_1|]$$

for any $x, x_1 \in Q_p$, $z \in S_n$

$$\int_{S_n} \sup_x \|f(x,v,z)\|_p \|z\|_p^{-\alpha-1} dz dt \le N_n|v|,$$

where M_n depends on $\max(|v|,|v_1|)$ and N_n depends on $|v|$ and

$$\sum_{n=-\infty}^{\infty} L_n p^{-n(\alpha+1)} \le L < \infty, \quad \sum_{n=-\infty}^{\infty} M_n p^{-n(\alpha+1)} \le B < \infty$$

and $\Sigma_{n=-\infty}^{\infty} N_n p^{-n(\alpha+1)} \le B < \infty$.

Let $v(t,x)$ be a scalar function on $[0,T] \times Q_p$ such that

$$|v(t,x) - v(t,y)| \le L_v(t)\|x-y\|_p,$$

$$\sup_{x \in Q_p} |v(t,x)| \le K_v(t) < \infty$$

with positive real valued functions $L_v(t), K_v(t)$ bounded on an interval $[0,T]$.

Consider a stochastic equation of the form

$$\xi(t) = x + \int_0^t \int_{Q_p} f(\xi(\theta), v(\theta, \xi(\theta)), z)\nu_\alpha(d\theta, dz). \tag{2.8}$$

Lemma 2.1. *Assume that* **C 1.1 – C.1.4** *and* **C 2.1** *hold. Then the solution* $\xi_{s,x,v}(t)$ *satisfies the following estimates*

$$E\|\xi_{s,x,v}(t) - \xi_{s,y,v}(t)\|_p \le \|x-y\|_p \tag{2.9}$$

$$E\|\xi_{s,x,v}(t) - \xi_{s,x,v_1}(t)\|_p \leq$$
$$e^{\int_0^t [L+BL_v(\tau)]d\tau} \int_0^t |K_v(\tau) - K_{v_1}(\tau)|d\tau, \quad (2.10)$$
$$\|\xi_{s,x,v}(t) - x\|_p < \infty. \quad (2.11)$$

Lemma 2.2. *Under the conditions of lemma 2.1 assume in addition that the function* φ *is bounded and Lipschitz continuous and*

$$sup_{x\in Q_p}|\phi(x)| \leq K_0,$$
$$|\phi(x) - \phi(y)| \leq L_0\|x - y\|_p, x, y \in S_n.$$

Then the function $u^v(s, x) = E[\varphi(\xi_{s,x,v}(T))]$ *satisfies the following estimates*

$$|u^v(s, x) - u^v(s, y)| \leq L_0\|x - y\|_p, \quad (2.12)$$

$$|u^v(s, x) - u^{v_1}(s, x)| \leq$$
$$L_0 e^{\int_s^t [L+BL_v(\tau)]d\tau} \int_s^t B|K_v(\tau) - K_{v_1}(\tau)|d\tau. \quad (2.13)$$
$$sup_{x\in Q_p}|u(s, x)| \leq C < \infty. \quad (2.14)$$

The proof obviously follows from lemma 2.1 due to properties of φ.

Lemma 2.3. *Under the conditions of lemma 2.2 there exist functions* $\alpha(s)$, $\beta(s)$ *bounded for* $s \in \Delta_1$. *In addition, for all* $t \in \Delta_1$ *if* $f, g \in \mathcal{L}$ *possess the properties* $\|v(s)\|_{\mathcal{L}} \leq \alpha(s)$ *and* $|v(s, x) - v(s, y)| \leq \beta(s)\|x - y\|$, *with bounded* $\alpha(s)$,$\beta(s)$ *for* $s \in [0, T]$ *then for* $s \in [0, T]$

$$\|u^v(s)\|_{\mathcal{L}} \leq \alpha(s), \quad |u^v(s, x) - u^v(s, y)| \leq \beta(s)\|x - y\|_p. \quad (2.15)$$

Theorem 2.1. *Under the conditions of lemma 2.2 there exists a unique solution to* $\xi(t), u(s,x)$ *of the system (2.3),(2.4) for* $0 \le s \le t \le T$. *The process* $\xi(t)$ *is a Markov process in* Q_p, *while* $u(s,x)$ *is a bounded and Lipschitz continuous scalar function.*

Proof. By lemma 2.3 we know that the mapping $\Phi(s,x,v) = E\varphi_0(\xi_{s,x,v}(t))$ acts in the space $\mathcal{L}$. Let

$$r^k(t-s,x) = |u^{k+1}(t-s,x) - u^k(t-s,x)|$$

and $\zeta^k(t-s) = sup_x r^k(t-s,x)$.

By the estimates of lemma 2.1 we have

$$r^k(t-s,x) \le$$

$$L_0 \int_s^t M\|u^k(t-\tau) - u^{k-1}(t-\tau)\|_{\mathcal{L}}^2 d\tau e^{\int_s^t \beta(t-\tau)d\tau}$$

and hence

$$\zeta^k(t-s) \le \delta^k \int_s^t \dots \int_s^{t_2} \|u^1(t-\tau_1) - u^0\| d\tau_1 \dots d\tau_k$$

holds with $\delta = L_0 e^{Q\int_s^t \beta(t-\tau)d\tau}$ and $Q = L_{(u_k,u_{k-1})}$. Notice that Q depends on sup-norm of functions $u_k(s), u_{k-1}(s)$ that are bounded for $t \in [0,T]$ due to estimates of Lemma 2.2. Since u^k are uniformly bounded by K_0 and

$$\|u^1(t-s,\cdot) - u^0(\cdot)\|_{\mathcal{L}} \le const < \infty,$$

we get

$$\|u^k(t-s,\cdot) - u^{k-1}(t-s,\cdot)\|_{\mathcal{L}} \le \frac{N^k}{k!} const$$

where $N = T\delta$. Putting $s = 0$ we obtain that the family $u^k(t,\cdot)$ uniformly converges to a limiting function $f(t,\cdot)$ for all $t \in [0,T]$. In addition, it is easy to check that

$u(s,x)$ is Lipschitz continuous in x. In fact by lemma 2.2 for each $s \in [0,T]$ we have

$$|u^k(s,x) - u^k(s,y)| \leq L_0 \|x-y\|$$

and the estimate is uniform in k.

To prove that the above constructed solution is unique we assume on the contrary that there exist two solutions $u_1(t,x)$, $u_2(t,x)$ to (2.3), (2.4) possessing the same initial data $u_1(0,x) = u_2(0,x) = u_0(x)$. It results from lemma 3.1 that there exists a constant C such that

$$\|u_1(t,\cdot) - u_2(t,\cdot)\|_{\mathcal{L}} \leq C \int_0^t \|u_1(\tau,\cdot) - u_2(\tau,\cdot)\|_{\mathcal{L}} d\tau$$

and hence by the Gronwall lemma

$$\|u_1(t,\cdot) - u_2(t,\cdot)\|_{\mathcal{L}} = 0.$$

By theorem 2.1 and theorem 1.1 we know that the process $\xi(t)$ satisfying (2.3) is a Markov process.

The generator of this Markov process and the generator of the evolution family $U^u(s)\varphi(x) = E\varphi(\xi^u_{s,x}(T))$ generated by this process, has the form

$$\mathcal{A}^u \varphi(x) = \lim_{t \to s} \frac{E\varphi(\xi^u_{s,x}(t)) - \varphi(x)}{t-s} =$$

$$\frac{p^\alpha - 1}{1 - p^{-\alpha-1}} \int_{Q_p} [\varphi(x + f(s,x,,u(s,x),z)) - \varphi(x)] \|z\|_p^{-\alpha-1} dz \tag{2.16}$$

Theorem 2.2 *Assume that* **C 1.1- C 1.4** *and* **C 2.1** *hold. Then there exists a unique solution of the Cauchy problem (2.1) on the interval* $[0,T]$.

3 Stochastic processes associated with linear systems of equations

Denote by c^* a transposed matrix to a matrix c. Consider the system of SDEs

$$\begin{cases} \xi(t) = & x + \int_s^t \int_{Q_p} f(\theta, \xi(\theta), z)\nu_\alpha(d\theta, dz), \\ \eta(t) = & h + \int_s^t c^*(t, \xi(\theta))\eta(\theta)d\theta + \\ & \int_s^t \int_{Q_p} \gamma^*(\theta, \xi(\theta), z)\eta(\theta)\nu(d\theta, dz), \end{cases} \tag{3.1}$$

where $x \in Q_p, h \in R^d,\ c(t,x), \gamma(t,x,z) \in R^d \times R^d$ are matrix valued functions defined on $[0,T] \times Q_p$ and $[0,T] \times Q_p \times Q_p$ respectively and $\eta(t) \in R^d$ is a vector valued jump process.

Let M^d denote the space of $d \times d$ matrices and $B(\Omega, M)$ denote the space of random variables valued in M. Denote by $\|h\|$ the norm of the vector $h \in R^d$. Assume that the following estimates hold.

C 3.1. $\gamma(\cdot, x, z) \in D([0,T], C_b(Q_p \times S_n, B(\Omega, M^d)))$ for every $n \in Z$ where C_b is the space of bounded continuous matrix functions with a natural sup norm.

C 3.2. There exists a positive constant C such that

$$\int_0^T \int_{Q_p} \sup_{x \in Q_p} \|\gamma(t,x,z)h\|_p^\gamma \|z\|^{-\alpha-1} dz dt < C\|h\|$$

for any $h \in R^d$ with probability 1.

C 3.3. For each $n \in Z$, there exists a constant $L_n > 0$ such that for all $t \in [0,T], \quad x, y \in Q_p, z \in S_n$ (where $S_n = \{x \in Q_p : \|x\|_p = p^n\}$) such that

$$\|\gamma(t,x,z)h - \gamma(t,y,z)h\| \le K_n \|x-y\|_p \|h\| \tag{3.2}$$

with probability 1 and

$$\sum_{n=-\infty}^{\infty} K_n p^{-n(\alpha+1)} \le C < \infty.$$

Theorem 3.1. *Let conditions* **C 1.1 - C 1.4**, **C 3.1 - C 3.4** *hold. Then there exists a unique solution to the system (3.1). The process $\eta(t) \in R^d$ in (3.1) possesses the following properties*

$$E\|\eta(t)\| \leq e^{C(t-s)}\|h\|$$

and defines a multiplicative operator functional of the Markov process $\xi(t) \in Q_p$ that is $\eta(t) = Z(t,s)h$ and $Z(t,s)h = Z(t,\tau)Z(\tau,s)h$.

Let us consider a vector function

$$g(s,x) = EZ^*(s,T)m(\xi(T)) \tag{3.3}$$

where $Z^*(s,T)$ is dual to $Z(s,T)$

$$\langle Z(T,s)h, m(x)\rangle = \langle h, Z^*(s,T)m(x)\rangle.$$

Denote by $U(s,t)$ a mapping acting in the space of bounded R^d-valued functions defined on $[0,T] \times Q_p$ by

$$U(s,t)m(x) = EZ^*(s,t)m(\xi(t)). \tag{3.4}$$

Theorem 3.2. *Under the conditions of theorem 3.1 the family $U(s,t)$ defined by (3.4) is an evolution family of bounded linear mappings acting in the space $C([0,T] \times Q_p; R^d)$. The generator of this evolution family is given by*

$$\mathcal{A}_1(t)m(x) = \lim_{s\to t} \frac{EZ^*(s,t)m(\xi(t)) - m(x)}{t-s} =$$

$$c(t,x)m(x)+$$

$$\frac{p^\alpha - 1}{1-p^{-\alpha-1}}[\int_{Q_p} \gamma(t,x,z)m(x+f(t,x,z))\|z\|_p^{-\alpha-1}dz+$$

$$\int_{Q_p}[m(x+f(t,x,z)) - m(x)]\|z\|_p^{-\alpha-1}dz]. \tag{3.5}$$

Theorem 3.3. *Under the conditions of theorem 3.1 the function $g(s,x)$ defined by (3.3) is a unique solution of the Cauchy problem*

$$\frac{\partial g(s,x)}{\partial s} + c(s,x)g(x)+$$

$$\frac{p^{\alpha}-1}{1-p^{-\alpha-1}}\int_{Q_p}[\gamma(s,x,z)[g(s,x+f(s,x,z))]\|z\|_p^{-\alpha-1}dz+$$

$$\frac{p^{\alpha}-1}{1-p^{-\alpha-1}}\int_{Q_p}[g(s,x+f(s,x,z))-g(s,x)]]\|z\|_p^{-\alpha-1}dz=0, \tag{3.6}$$

$$g(T,x)=m(x).$$

4 Q_p-analogue of the KPP equation

In this section we combine the results of the previous sections to construct a probabilistic representation of the solution for the following Cauchy problem

$$\frac{\partial v(t,x)}{\partial t} = \int_{Q_p}[v(t,x+f(x,z))-v(t,x)]]\|z\|_p^{-\alpha-1}dz+$$

$$c(x,v(t,x))v(t,x), \quad v(0,x)=v_0(x). \tag{4.1}$$

Recall that in the Euclidian case $x\in R^d$, $f(t,x)\in R^1, t\in [0,T]$ the probabilistic approach to the equation

$$\frac{\partial u}{\partial t} = \frac{\sigma^2}{2}\Delta u + c(u)u, \quad u(0)=u_0(x) \tag{4.2}$$

called KPP equation was developed by Freidlin [5] and extended by Dalecky and Belopolskaya [2], [6]. In particular it was proved that can reduce (4.2) to the following integral equation

$$u(t,x) = E\left[e^{\int_0^t c(w_x(\tau),u(t-\tau,w_x(\tau)))d\tau} u_0(w_x(t))\right], \quad (4.3)$$

where $w_x(t) = x + \sigma w(t)$ and $w(t) \in R^d$ is a Wiener process and σ is a constant. Finally if $c(x,v)$ is a bounded function in x with polynomial growth in v and possessing two continuous bounded derivatives in x and f then the solution of (4.3) defines a classical solution to (4.2).

We extend this result to the case $x \in Q_p, v(t,x) \in R^1$. To this end we assume that the function $c(x,g) \in R^1$ defined on $Q_p \times R^1$ satisfies the following condition **C 4.1**

$$|c(x,g| \leq K[1 + |g|^r]$$

$$|c(x,g) - c(y,\tilde{g})| \leq L_1\|x-y\|_p + M(g,\tilde{g})|g-\tilde{g}|$$

where K, L are absolute constants and $M(g,\tilde{g}) < \infty$ depends on $max(g,\tilde{g})$, $g, \tilde{g} \in R^{1\text{`}}$.

To construct a probabilistic counterpart of (4.1) we consider a stochastic equation

$$d\xi(\tau) = \int_{Q_p} f(\xi(\tau),z)\nu_\alpha(d\tau,dz), \quad \xi(s) = x \in Q_p, \quad (4.4)$$

and a function $v_s(t,x)$ of the form

$$v_s(t,x) = E_{s,x}[e^{\int_s^t c(\xi(\tau),v(t-\tau,\xi(\tau)))d\tau} v_0(\xi(t))] \quad (4.5)$$

Similar to the previous section we start with deriving some apriori estimates. Namely we say that condition **C 4.2** holds if a scalar function $g(t,x)$ defined on $[0,T]\times Q_p$ satisfies estimates

$$\sup_{x\in Q_p} |g(t,x)| \leq \kappa(s), \quad |g(s,x) - g(s,y| \leq \zeta(s)\|x-y\|_p$$

with bounded scalar functions $\kappa(s), \zeta(s)$ defined on $[0, T]$ and set

$$v^g(t, x) = E_{s,x}[e^{\int_0^t c(\xi(\tau), g(t-\tau, \xi(\tau)))d\tau} v_0(\xi(t))] \qquad (4.6)$$

Lemma 4.1*Assume conditions* **C 1.1-C 1.4** *and* **C 4.1-C 4.2** *to be hold and let $v_0(x)$ satisfy the estimates*

$$\sup_{x \in Q_p} |v_0(x)| \leq K_0, \quad |v_0(x) - v_0(y)| \leq L_0 \|x - y\|_p.$$

Then the function $v^g(t, x)$ given by (4.6) satisfies the estimates

$$\sup_{x \in Q_p} |v^g(t, x)| \leq K_0 e^{\int_0^t K[1 + K_g^r(t-\tau)]d\tau}$$

$$|v^g(t, x) - v^g(t, y)| \leq [L_0 e^{\int_0^t K[1 + K_g^r(t-\tau)]d\tau} +$$
$$K_0 e^{\int_0^t [1 + K_g^r(t-\tau)]d\tau} \int_0^t [L_1 + M L_g(t - \tau)]d\tau] \|x - y\|_p$$

Proof. The proof of the first estimate immediately deduced from the conditions of the lemma. Denote $\eta_x(t) = e^{\int_0^t c(\xi(\tau), g(t-\tau, \xi(\tau)))}$. Obviously $\eta_x(t)$ solves the equation

$$\eta_x(t) = 1 + \int_0^t c(\xi_x(\tau), g(t - \tau, \xi_x(\tau))\eta_x(\tau)d\tau.$$

Let us check that

$$|v^g(t, x) - v^g(t, y)| = E|\eta_x(t)v_0(\xi_x(t)) - \eta_y(t)v_0(\xi_y(t))| \leq$$

$$E|\eta_x(t) - \eta_y(t)||v_0(\xi_x(t))| + E|\eta_y(t)|v_0(\xi_x(t)) - v_0(\xi_y(t))| \leq$$
$$[K_0 e^{\int_0^t [L_1 + M L_g(t-\tau)]d\tau} + e^{\int_0^t [1 + K_g(t-\tau)]d\tau}]\|x - y\|_p.$$

Lemma 4.1*Let conditions of lemma 4.1 hold. Then there exist positive functions $\alpha(t), \beta(t)$ such that if*

$\sup_{x\in Q_p}|g(t,x)| \leq \alpha(t)$ *and* $|g(t,x)-g(t,y) \leq \beta(t)\|x-y\|_p$ *on a certain interval* $[0,T]$ *then*

$$\sup_{x\in Q_p}|v^g(t,x)| \leq \alpha(t)$$

and

$$|v^g(t,x)-v^g(t,y)| \leq \beta(t)\|x-y\|_p$$

on the same interval.

Proof. Choose $\alpha(t)$ as a solution of the integral equation

$$\alpha(t-s) = K_0 e^{\int_s^t [K+\alpha(t-\tau)]d\tau}. \tag{4.7}$$

One can easily check that if $g(t,x) \leq \alpha(t)$ then

$$\sup_{x\in Q_p} v^g(t,x)| \leq \alpha(t).$$

Since $\alpha(t)$ given by (4.7) satisfies the ordinary differential equation

$$\frac{d\alpha(t-s)}{ds} = -[K+\alpha(t-s)]\alpha(t-s), \quad \alpha(0) = K_0, \tag{4.8}$$

we can easily check that

$$\alpha(t) = \frac{K_0 K e^{Kt}}{K+K_0-K_0e^{Kt}} \tag{4.9}$$

is the solution to (4.8). In addition the function $\alpha(t)$ defined by (4.9) is bounded on the interval $[0,T_1]$ where

$$T_1 < \frac{1}{K}\ln[1+\frac{K}{K_0}] \tag{4.10}$$

In a similar way we can prove that the function $\beta(t)$ that solves the integral equation

$$\beta(t) = 2K_0 e^{\int_0^t [L_1+M\beta(t-\tau)]d\tau}$$

and has the form

$$\beta(t) = \frac{2K_0L_1}{L_1 + 2K_0 - 2K_0e^{L_1t}} \tag{4.11}$$

and it is the required function which is bounded on the interval

$$T_2 < \frac{1}{L_1}\ln[1 + \frac{L_1}{2K_0}] \tag{4.12}$$

Theorem 4.1. *Assume conditions* **C 1.1-C 1.4** *to be hold and let* **C 4.1- C 4.2** *be hold. Then there exists a unique solution of the system (4.4), (4.5). In addition the function $v(t,x)$ determined by (4.5) with $s = 0$ is a unique solution of (4.1).*

Proof. Note first that under the conditions of the theorem there exists the unique solution $\xi(t)$ of the stochastic equation (4.4). Next we consider the integral equation (4.5) and check that there exists a unique solution of this equation. To this end we construct a family of successive approximations

$$v_s^0(t,x) = v_0(x), \tag{4.13}$$

$$v_s^{n+1}(t,x) = E_{s,x}[e^{\int_s^t c(\xi(\tau),v_s^n(t-\tau,\xi(\tau)))d\tau}v_0(\xi(t))]. \tag{4.14}$$

First we derive the estimate for $\kappa^n(s) = \sup_{x\in Q_p}|v_s^n(t,x)|$. By reasons similar to that used in the previous chapters we can derive the following apriori estimates

$$\kappa_s^{n+1}(t) \le K_0e^{\int_0^t K[1+(\kappa_s^n)^r(t-\tau)]d\tau}$$

that allows to show that the function $\alpha(t-s)$ of the form (4.9) possesses the property $\sup_x|v_0^n(t,x)| < \alpha(t)$. Hence the function $\alpha(t)$ is a majorizing function for the family $v_0^n(t,x)$.

Besides we have to estimate the function $\zeta^n(t-s)$ such that $|v^n(t,x) - v^n(t,y)| \le \zeta^n(t-s)\|x-y\|_p$. By lemma 4.2 we deduce a positive function $\beta(t)$ of the form (4.11) play a part of a majorising function for $\zeta^n(t)$ on the interval $[0, T_2]$ T_1 and T_2 are defined by (4.10), (4.12).

Finally we prove that the family $v^n(t,x)$ uniformly in $x \in Q_p$ converges to a limit $v(t,x)$ on the interval $[0, T_3]$ where $T_3 < \min(T_1, T_2)$.

To fulfill the proof of the theorem 4.1 we have to verify that the function $v(t,x)$ defined by (4.5) satisfies the Cauchy problem (4.1). But as soon as the function $v(t,x)$ is defined and possesses the required properties we are in the framework of section 1.

References

[1] A. Kochubei, Pseudo-differential equations and stochastics over non-archimedian fields. Marcel Dekker Inc. 2001.

[2] Ya. Belopolskaya, Yu. Dalecky *Investigation of the Cauchy problem for systems of quasilinear equations via Markov processes* . Izv VUZ Matematika. N 12 (1978) 6-17.

[3] Ya. Belopolskaya, *Systems of quasilinear integro-differential equations and Markov processes associated with them*, Probabilistic distributions in infinite dimensional spaces. Nauk. Dumka, (1978) p. 5-21.

[4] Ya. Belopolskaya, *On stochastic equations with unbounded coefficients for jump processes*, Lecture

Notes in Control and information Theory, **25**, (1980), 243–254.

[5] M. Freidlin *Functional integration and partial differential equations*. Princeton Univ. Press 1985.

[6] Ya.I. Belopolskaya, Yu.L. Dalecky*Stochastic equations and differential geometry*. Kluwer (1990).