p-Adic refinable functions and MRA-based wavelets

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Haar basis in real analysis

$\{\psi_{jk}\} \quad \psi_{jk}(x) = 2^{j/2}\psi(2^j x + k),\ j, k \in \mathbb{Z},$

$$\psi(x) = \begin{cases} 1, & 0 < x < 1/2, \\ -1, & 1/2 < x < 1, \\ 0, & otherwise. \end{cases} \qquad \textit{A. Haar, 1910}$$

p-adic analog of Haar basis

$\{\psi_{ja}^{(\nu)}\} \ \ \psi_{ja}^{(\nu)} = p^{j/2}\psi^{(\nu)}(p^{-j} \cdot - a), a \in I_p, j \in \mathbb{Z},$

$\psi^{(\nu)}(x) = \chi_p(\frac{\nu}{p}x)\Omega(|x|_p),\ \nu = 1, \ldots, p-1,$

$$\chi_p(u) = e^{2\pi i \{u\}_p},\ \Omega(t) = \begin{cases} 1, & 0 \le x \le 1, \\ 0, & otherwise. \end{cases}$$

$$p = 2: \quad \psi(x) = \chi_2(\tfrac{1}{2}x)\Omega(|x|_2) = \begin{cases} 1, & |x|_2 \le 1/2, \\ -1, & 1/2 < |x|_2 \le 1, \\ 0, & otherwise. \end{cases}$$

S.Kozyrev. 2002

Haar p-adic wavelets are eigenfunctions of the p-adic pseudo-differential operators

Wavelet bases in real analysis

$\{\psi_{jk}\} \quad \psi_{jk}(x) = 2^{j/2}\psi(2^j x + k),\ j, k \in \mathbb{Z},$

A general scheme for the construction of wavelet bases was developed about 1990. This scheme is based on the notion of **multiresolution analysis**

Y. Meyer and S. Mallat

Wavelet bases in p-adic analysis

$\{\psi_{ja}^{(\nu)}\} \ \ \psi_{ja}^{(\nu)} = p^{j/2}\psi^{(\nu)}(p^{-j}\cdot - a), a \in I_p, j \in \mathbb{Z},$

$I_p = \{a = \frac{q}{p^s},\ q = 0, \dots, p^s - 1,\ s = 1, 2, \dots\}$

We have a "natural" decomposition of $\mathbb{Q}_p$ to a union of mutually disjoint discs: $\mathbb{Q}_p = \cup_{a \in I_p} B_0(a)$.

So, I_p is a *"natural" set of shifts* for $\mathbb{Q}_p$.

Conjecture: MRA theory can not be constructed in p-adic analysis (*J.J.Benedetto, 2004*)

Definition *A collection of closed spaces* $V_j \subset L^2(\mathbb{Q}_p)$, $j \in \mathbb{Z}$, *is called a* **multiresolution analysis** (**MRA**) *in* $L^2(\mathbb{Q}_p)$ *if the following axioms hold*

(a) $V_j \subset V_{j+1}$ *for all* $j \in \mathbb{Z}$*;*

(b) $\bigcup_{j\in\mathbb{Z}} V_j$ *is dense in* $L^2(\mathbb{Q}_p)$*;*

(c) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$*;*

(d) $f(\cdot) \in V_j \Longleftrightarrow f(p^{-1}\cdot) \in V_{j+1}$ *for all* $j \in \mathbb{Z}$*;*

(e) there exists a function $\varphi \in V_0$ **(scaling function)** *such that the system* $\{\varphi(\cdot - a), a \in I_p\}$ *is an orthonormal basis for* V_0.

$\{p^{j/2}\varphi(p^{-j}\cdot - a), a \in I_p\}$ ONB for V_j, $j \in \mathbb{Z}$.

$W_j = V_{j+1} \ominus V_j, \qquad j \in \mathbb{Z},$

$f \in W_j \Longleftrightarrow f(p^{-1}\cdot) \in W_{j+1}, \quad j \in \mathbb{Z},$

$W_j \perp W_k, j \neq k,$ W_j **wavelet spaces**

$$\left.\begin{array}{l} V_{j+l} = V_j \oplus W_j \oplus W_{j+1} \oplus \ldots \oplus W_{j+l-1} \\ \text{axioms (b) and (c)} \end{array}\right\} \Longrightarrow \bigoplus_{j\in\mathbb{Z}} W_j = L^2(\mathbb{Q}_p)$$

$\psi^{(\nu)} \in W_0$ **(wavelet functions)**.

$\{\psi^{(\nu)}(\cdot - a), a \in I_p, \nu \in A\}$ ONB for W_0

$\{p^{j/2}\psi^{(\nu)}(p^{-j}\cdot - a), a \in I_p, \nu \in A, j \in \mathbb{Z}\}$ ONB $L^2(\mathbb{Q}_p)$;

Refinement equation

Let φ be a scaling function for a MRA.

$$\left.\begin{array}{ll} \text{axiom (e)}: & \varphi \in V_0 \\ \text{axiom (a)}: & V_0 \subset V_1 \end{array}\right\} \Longrightarrow \varphi \in V_1$$

Refinement equation

$$\varphi = \sum_{a \in I_p} \alpha_a \varphi(p^{-1} \cdot - a), \quad \alpha_a \in \mathbb{C},$$

is **necessary** for $V_0 \subset V_1$, but not **sufficient**.

Since

$$B_0(0) = B_{-1}(0) \cup \big(\bigcup_{r=1}^{p-1} B_{-1}(r) \big),$$

where $B_s(a) = \{x : |x - a|_p \le p^s\}$, we have

$$\varphi(x) = \sum_{r=0}^{p-1} \varphi\left(\frac{1}{p}x - \frac{r}{p} \right), \quad x \in \mathbb{Q}_p,$$

whose solution is $\varphi(x) = \Omega(|x|_p)$ is the characteristic function of the unit disk $B_0(0)$

Conjecture: this is a "natural" refinement equation for the p-adic Haar MRA. (*A.Krennikov, V.Shelkovich 2006*)

Construction of refinable functions

Now we are going to study p-adic refinement equations

$$\varphi(x) = \sum_{k=0}^{p^s-1} \beta_k \varphi\left(\frac{1}{p}x - \frac{k}{p^s}\right)$$

and their solutions **(refinable functions)**.

$$\widehat{\varphi}(\xi) = m_0\left(\frac{\xi}{p^{s-1}}\right)\widehat{\varphi}(p\xi),$$

where

$$m_0(\xi) = \frac{1}{p}\sum_{k=0}^{p^s-1} \beta_k \chi_p(k\xi)$$

is a trigonometric polynomial of order $p^s - 1$ (**mask**).

If $\widehat{\varphi}(0) \neq 0$, then $m_0(0) = 1$.

$$V_j = \overline{\mathrm{span}\{\varphi(p^{-j}\cdot - a) : a \in I_p\}}, \quad j \in \mathbb{Z}.$$

Theorem 1 *If φ is a refinable function such that $\mathrm{supp}\,\widehat{\varphi} \subset B_0(0)$ and the system $\{\varphi(x-a) : a \in I_p\}$ is orthonormal, then axiom (a) holds for the spaces V_j generated by φ.*

Theorem 2 *Let* $\varphi \in L^2(\mathbb{Q}_p)$, $\operatorname{supp}\widehat{\varphi} \subset B_0(0)$ *and the system* $\{\varphi(x-a) : a \in I_p\}$ *be orthonormal. Axiom* (b) *holds for the spaces* V_j *generated by* φ *(i.e.,* $\overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{Q}_p)$*) if and only if*

$$\bigcup_{j\in\mathbb{Z}} \operatorname{supp}\widehat{\varphi}(p^j\cdot) = \mathbb{Q}_p.$$

Theorem 3 *If* $\varphi \in L^2(\mathbb{Q}_p)$ *and* $\{\varphi(x-a) : a \in I_p\}$ *is an orthonormal system, then axiom* (c) *holds for the spaces* V_j *generated by* φ (i.e., $\cap_{j\in\mathbb{Z}} V_j = \{0\}$).

Theorem 4 *Let* φ *be a refinable function such that* $\operatorname{supp}\widehat{\varphi} \subset B_0(0)$. *If* $|\widehat{\varphi}(\xi)| = 1$ *for all* $\xi \in B_0(0)$ *then the system* $\{\varphi(x-a) : a \in I_p\}$ *is orthonormal.*

Proposition 5 *If $\varphi \in L^2(\mathbb{Q}_p)$ is a refinable function satisfying*

$$\widehat{\varphi}(\xi) = m_0\left(\frac{\xi}{p^{s-1}}\right)\widehat{\varphi}(p\xi),$$

$\widehat{\varphi}(\xi)$ is continuous at the point 0 and $\widehat{\varphi}(0) \neq 0$, then

$$\widehat{\varphi}(\xi) = \widehat{\varphi}(0)\prod_{j=1}^{\infty} m_0\left(\frac{\xi}{p^{s-j}}\right). \tag{1}$$

Proposition 6 *If $\widehat{\varphi}$ is defined by (1), where m_0 is a trigonometric polynomial and $m_0(0) = 1$, then φ is a refinable locally-constant function.*

Proposition 7 *Let $\widehat{\varphi}$ be defined by (1), where m_0 is a trigonometric polynomial of order $p^s - 1$. If $m_0(0) = 1$, $m_0(\frac{k}{p^s}) = 0$ for all $k = 1, \dots, p^s - 1$ which are not divisible by p, then $\operatorname{supp}\widehat{\varphi} \subset B_0(0)$, $\widehat{\varphi} \in L^2(\mathbb{Q}_p)$. If, furthermore, $|m_0(\frac{k}{p^s})| = 1$ for all $k = 1, \dots, p^s - 1$ which are divisible by p, then $|\widehat{\varphi}(x)| = |\widehat{\varphi}(0)|$ for any $x \in B_0(0)$.*

Due to Theorems 1-4, the refinable functions with masks satisfying the hypotheses of Proposition 7 generate MRAs.

Theorem 8 *Let* $\widehat{\varphi}$ *be defined by* (1)*, where* m_0 *is a trigonometric polynomial of order* p^s-1*. If the system* $\{\varphi(x-a) : a \in I_p\}$ *is orthonormal,* $\operatorname{supp}\widehat{\varphi} \subset B_0(0)$*, then* $|m_0(\frac{k}{p^s})| = 0$ *whenever* k *is not divisible by* p*, and* $|m_0(\frac{k}{p^s})| = 1$ *whenever* k *is divisible by* p*,* $k = 1, \ldots, p^s - 1$*.*

Conjecture: *it does not exist compactly supported refinable functions with mutually orthogonal shifts* $\{\varphi(x-a) : a \in I_p\}$ *whose Fourier transform is not supported in* $B_0(0)$*.*

Example $p = 2$, $s = 3$,

$m_0(1/4) = m_0(3/8) = m_0(7/16) = m_0(15/16) = 0$.

$\operatorname{supp}\widehat{\varphi} \subset B_1(0)$, $\operatorname{supp}\widehat{\varphi} \not\subset B_0(0)$, $\widehat{\varphi}(\frac{1}{2}) = \widehat{\varphi}(\frac{3}{2}) = \widehat{\varphi}(\frac{5}{2}) = \widehat{\varphi}(\frac{9}{2}) = \widehat{\varphi}(\frac{11}{2}) = \widehat{\varphi}(\frac{13}{2}) = \widehat{\varphi}(1) = \widehat{\varphi}(5) = 0$.

Axiom (a) will be fulfilled whenever

$$\varphi\left(x - \frac{k}{4}\right) = \sum_{r=0}^{7} \gamma_{kr}\varphi\left(\frac{1}{2}x - \frac{r}{8}\right), \quad k = 1, 2, 3,$$

which is equivalent to

$$\widehat{\varphi}(\xi)\chi_2\left(\frac{k\xi}{4}\right) = m_k\left(\frac{\xi}{4}\right)\widehat{\varphi}(2\xi), \quad k = 1, 2, 3,$$

where $m_k(\xi) = \frac{1}{2}\sum_{r=0}^{7}\gamma_{k,r}\chi_2(r\xi)$.

$$\widehat{\varphi}(8\xi)(m_0(\xi)\chi_2(k\xi)) - m_k(\xi)) = 0, \quad k = 1, 2, 3,$$

These equalities will be fulfilled for any $\xi \in \mathbb{Q}_2$ whenever they are fulfilled for $\xi = l/16$, $l = 0, 4, 6, 7, 8, 12, 14, 15$.

Proposition 9 *For any refinable function whose Fourier transform is in $B_1(0)$ but not in $B_0(0)$ the shift system $\{\varphi(x - a) : a \in I_2\}$ is not orthogonal.*

Construction of wavelet bases

$$\varphi(x) = \sum_{k=0}^{p^s-1} \beta_k \varphi\left(\frac{1}{p}x - \frac{k}{p^s}\right)$$

$W_0 = V_1 \ominus V_0, \qquad j \in \mathbb{Z},$
$\psi^{(\nu)} \in W_0$, $\nu = 1, \ldots, p-1$
$\{\psi^{(\nu)}(x-a), a \in \ I_p, \nu \in A\}$ ONB for W_0

$$\psi^{(\nu)}(x) = \sum_{k=0}^{p^s-1} \gamma_{\nu k} \varphi\left(\frac{1}{p}x - \frac{k}{p^s}\right)$$

$(\psi^{(\nu)}, \varphi(\cdot - a)) = 0,$
$(\psi^{(\nu)}, \psi^{(\mu)}(\cdot - a)) = \delta_{\nu\mu}\delta_{0a}, \quad \nu, \mu = 1, \ldots, p-1,$
for $a = 0, \frac{1}{p^{s-1}}, \ldots, \frac{p^{s-1}-1}{p^{s-1}}$.

$$B = \tfrac{1}{\sqrt{p}}(\beta_0, \ldots, \beta_{p^s-1})^T, S = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}.$$

Find $G_\nu = \frac{1}{\sqrt{p}}(\gamma_{\nu 0}, \ldots, \gamma_{\nu, p^s-1})^T$, $\nu = 1, \ldots, p-1$, such that the matrix

$$(S^0 B, \ldots, S^r B, S^0 G_1, \ldots, S^r G_1, \ldots, S^0 G_{p-1}, \ldots, S^r G_{p-1})$$

is unitary ($r = p^{s-1} - 1$).

Example $s = 1$

$m_0(0) = 1$, $m_0(\frac{k}{p}) = 0$, $k = 1, \dots, p-1$

$m_0(\xi) = \frac{1}{p}\sum_{k=0}^{p-1}\chi_p(k\xi)$ $(B = \frac{1}{\sqrt{p}}(1, \dots, 1)^T)$

$\varphi(x) = \sum_{r=0}^{p-1}\varphi\left(\frac{1}{p}x - \frac{r}{p}\right)$,

$\{\frac{1}{\sqrt{p}}e^{2\pi ikl}\}_{k,l=0,\dots,p-1} = (B, G_1, \dots, G_{p-1})$

$\psi^{(\nu)}(x) = \chi_p(\frac{\nu}{p}x)\Omega(|x|_p)$, $\nu = 1, \dots, p-1$.

Theorem 10 *Let $p = 2$. The function*

$$\psi(x) = \sum_{k=0}^{2^s-1}\alpha_k\psi^{(1)}\left(x - \frac{k}{2^s}\right),$$

is a wavelet function for the Haar MRA iff

$$\alpha_k = 2^{-s}(-1)^k\sum_{r=0}^{2^s-1}\gamma_r e^{-i\pi\frac{2r+1}{2^S}k}, \quad k = 0, \dots, 2^s - 1,$$

where $\gamma_r \in \mathbb{C}$, $|\gamma_r| = 1$.

Example $s = 2$, $p = 3$.
$m_0(\frac{k}{9}) = 0$ if k is not divisible by 3,
$m_0(0) = 1$, $m_0(\frac{1}{3}) = m_0(\frac{2}{3}) = -1$.

$m_0(z) = 3^{-2}(-1+2z+2z^2-z^3+2z^4+2z^5-z^6+2z^7+2z^8)$,
where $z = e^{2\pi i\xi}$

$$\widehat{\varphi}(\xi) = \begin{cases} 1, & |\xi|_p \le \frac{1}{3}, \\ -1, & |\xi - 1|_p \le \frac{1}{3}, \\ -1, & |\xi - 2|_p \le \frac{1}{3}, \\ 0, & |\xi|_p \ge 3. \end{cases}$$

$$\varphi(x) = \begin{cases} -\frac{1}{3}, & |x|_p \le \frac{1}{3} \\ \frac{2}{3}, & |x - \frac{1}{3}|_p \le 1 \\ \frac{2}{3}, & |x - \frac{2}{3}|_p \le 1 \\ 0, & |x|_p \ge 9 \end{cases}$$

$$= \frac{1}{3}\Omega(|3x|_p)(1 - e^{2\pi i\{x\}_3} - e^{4\pi i\{x\}_3}).$$

$$B = \frac{1}{3\sqrt{3}}(-1, 2, 2, -1, 2, 2, -1, 2, 2)^T,$$

$$(B, SB, S^2B, G_1, SG_1, S^2G_1, G_2, SG_2, S^2G_2)$$

$$G_1 = \frac{1}{\sqrt{3}}(1, 0, 0, -1, 0, 0, 0, 0, 0)^T$$
$$G_2 = \frac{1}{\sqrt{3}}(1, 0, 0, 1, 0, 0, -2, 0, 0)^T$$

$$\psi^{(1)} = \sqrt{\frac{3}{2}}(\varphi(\tfrac{x}{3}) - \varphi(\tfrac{x}{3} - \tfrac{1}{3})),$$
$$\psi^{(2)} = \frac{1}{\sqrt{2}}(\varphi(\tfrac{x}{3}) + \varphi(\tfrac{x}{3} - \tfrac{1}{3}) - 2\varphi(\tfrac{x}{3} - \tfrac{2}{3}))$$

$$\psi^{(1)}(x) = \begin{cases} -\sqrt{\frac{3}{2}}, & |x|_p \le \frac{1}{3}, \\ \sqrt{\frac{3}{2}}, & |x-1|_p \le \frac{1}{3}, \\ 0, & |x-2|_p \le \frac{1}{3}, \\ 0, & |x|_p \ge 3; \end{cases}$$

$$\psi^{(2)}(x) = \begin{cases} -\frac{1}{\sqrt{2}}, & |x|_p \le \frac{1}{3}, \\ -\frac{1}{\sqrt{2}}, & |x-1|_p \le \frac{1}{3}, \\ \sqrt{2}, & |x-2|_p \le \frac{1}{3}, \\ 0, & |x|_p \ge 3. \end{cases}$$