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Boundary-value problems for open and closed p-adic strings theory

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1. Introduction

For describtion tachyon dynamics of open-closed p-adic string the following non-linear pseudo-differential equations of motion

$$p^{-\frac{1}{4}\Box}\Psi = \Psi^{p^2} + \lambda^2\Psi^{p(p-1)/2}\Big(\Phi^{p+1} - 1\Big), \qquad (1.1a)$$

$$p^{-\frac{1}{2}\Box}\Phi = \Phi^p\Psi^{p(p-1)/2} \qquad (1.1b)$$

has been suggested (Brekke, Freund, 1993; Moeller, Schnabl, 2004). Here $\Psi(t)$ and $\Phi(t)$ are tachyon fields for closed and open strings resp., $\Box = -\partial_t^2 + \nabla_x^2$ is d-dimentional d'Alembertian, $\lambda = \frac{h}{g}$, where h and g are closed and open strings coupling constantas resp., p is a prime, $p = 2, 3, 5, \ldots$. (In what follows, p is assumed to be an integer greater then one.)

We consider one-dimetional case $d = 1$ and $\lambda = 0$.

$$\Psi^{p^2}(t) = (p^{1/4\partial_t^2}\Psi)(t), \quad t \in \mathbb{R}, \qquad (1.2a)$$

$$\Phi^p(t)\Psi^{p(p-1)/2}(t) = (p^{1/2\partial_t^2}\Phi)(t), \quad t \in \mathbb{R}, \qquad (1.2b)$$

The new class of non-linear Eqs (1.1) contains formaly an infinite number of derivatives reflecting a non-local interaction in string field theory. They involve pseudo-differential operators within the symbols $p^{-\xi^2/4}$ and $p^{-\xi^2/2}$. After changing the arguments of the fields

$$\psi(t) = \Psi(t\sqrt{\log p}), \quad \varphi(t) = \Phi(t\sqrt{\log p}),$$

the system of Eqs (1.2) takes the form

$$\psi^{p^2}(t) = (e^{1/4\partial_t^2}\psi)(t), \quad t \in \mathbb{R}, \qquad (1.3a)$$

$$\varphi^p(t)\psi^{p(p-1)/2}(t) = (e^{1/2\partial_t^2}\varphi)(t), \quad t \in \mathbb{R}, \qquad (1.3b)$$

where the integral operators $e^{x\partial_t^2}$ are defined by the formular

$$(e^{x\partial_t^2}f)(t) = \frac{1}{\sqrt{4\pi x}}\int_{-\infty}^{\infty}\exp\Big[-\frac{(t-\tau)^2}{4x}\Big]f(\tau)d\tau, \quad x > 0. \qquad (1.4)$$

We point out that the veriables x and t in Eq.(1.4) have been interchanged compared with the classical heat conduction operator. The family of operators $e^{x\partial_t^2}, x > 0$ form a semigroup

$$e^{x\partial_t^2} e^{y\partial_t^2} = e^{(x+y)\partial_t^2}, \quad x > 0, y > 0.$$

Eq.(1.3b) for $\psi = 1$ defines equation of motion for open string

$$\varphi^p = (e^{1/2\partial_t^2}\varphi)(t), \quad t \in \mathbb{R}. \tag{1.5}$$

System of Eqs (1.3) has the following trivial solutions (ψ, φ) (vacuums):

$$\begin{gathered}(0,0),(1,1),(1,0) \quad \forall p; \quad (1,-1),(-1,0) \quad p-\text{ odd };\\ (-1,1),(-1,-1) \quad p = 4n+1.\end{gathered} \tag{1.6}$$

For Eq.(1.3a) (closed string) in accordance with the vacuum solutions (1.6), we set the boundary conditions

$$\lim_{t\to-\infty} \psi(t) = \lim_{t\to\infty} \psi(t) = 1. \tag{1.7}$$

– boundary-value problems (b.-v.ps) (1.3a)-(1.7). For Eq.(1.5) (open string) -

$$\lim_{t\to\infty} \varphi(t) = 1, \quad \lim_{t\to-\infty} \varphi(t) = \begin{cases} -1, & p-\text{ odd },\\ 0, & p-\text{ even}\end{cases} \tag{1.8}$$

– b.-v.p. (1.5)–(1.8). Only real non-trivial solutions are physically intresting, and we consider only such solutions in what follows.

The posed b.-v.ps and their generalizations are of intrest not only for p-adic string theory but also for cosmology. Many physicists and mathematicians widely applying computer techniques made a lot of theoretical contributions to these b.-v.ps., among them Witten, Gross, Erler, Freund, Sen, Brekke, Moeller, Schnabl, Green, Schwarz, Frampton, Okada, Calgagni, Zwiebach, Ghoshal, Minahan, Barnaby, Coletti, Sigalov, Biswas, Cline,... As for Russian school, the first is I.Ja.Aref'eva and her many pupils – Koshelev, Joukovskaya, Zubarev,..., and also Volovich, Prokhorenko, Volovich-jr., Vladimirov.

We consider the posed problems in the class of real bounded functions, different from vacuums. The first question is existence or non-existens solutions. If they exist, we present the following topics:

- À priori properties and estimates of solutions.
- Hermit-series expantion of solutions.
- Reducing to a b.-v.p. for the heat equation.
- Tchebyshev-series expantion of periodic solutions.
- Application the Gauss quadrature formulas to approximate the equation.

For all b.-v.ps posed the first results are (Vladimirov, Volovich-jr., 2004):

Theorem 1. *If a solution belongs to the algebra of real tempered distributions* $\widetilde{\mathcal{S}'_+}$*, then it is a vacuum.*

Here $\widetilde{f}$ denotes the Fourier-transform of f and $\mathcal{S}'_+$ are tempered distributions with support in $\mathbb{R}_+$.

Theorem 2. If a solution is a non-negative bounded function, then it is a vacuum.

We consider more general equation

$$\varphi^p(t) = (e^{x_0 \partial_t^2} \varphi)(t), \quad t \in \mathbb{R}, \quad p \geq 2 \text{ integer }, \quad x_0 > 0 \tag{1.9}$$

with corresponding boundary conditions. It is equivalent to the non-linear b.-v.p. for the heat equation for *interpolaiting function* $u(x,t)$ (Vladimirov, 2005)

$$\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial t^2}, \quad 0 < x \leq x_0, t \in \mathbb{R}, \tag{1.10}$$

$$u(0,t) = \varphi(t), \quad u(x_0,t) = \varphi^p(t), \quad t \in \mathbb{R} \tag{1.11}$$

with the same boundary conditios. By this method many results has been obtained for b.-v.ps posed.

2. Closed string

For closed string the the Eq.(1.3a) is

$$\psi^{p^2}(t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(\tau-\tau)^2} \psi(\tau) d\tau, \quad t \in \mathbb{R} \tag{2.1}$$

with boundary conditions (1.7).

For even p there is no even continuous monotonic increasing solutions for $t > 0$ (Moeller, Zwiebach, 2002; Vladimirov); nevertheless, it is possible that piece-wise continuous solutions may exists, but the existens of such solutions do not jet proved. For $p = 3$, using numerical methods on computers, I.Ya.Aref'eva showed the existence (at least practically) even continuous solutions with two (simple) zeros.

Eq.(2.1) is equivalent to the non-linear integral equation

$$\psi^{p^2}(t) = \int_{-\infty}^{\infty} \mathcal{K}_{1/4}(t-\tau)[p^2\psi(\tau) - \psi^{p^2}(\tau)] d\tau, \quad t \in \mathbb{R}, \tag{2.2}$$

where the kernel $\mathcal{K}_{1/4}(t)$ is real continuous, positive and positive-definite function from $\mathcal{L}_1(\mathbb{R})$, defined by the formula

$$\mathcal{K}_{1/4}(t) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi n} p^{2n}} e^{-\frac{t^2}{n}}, \int_{-\infty}^{\infty} \mathcal{K}_{1/4}(t) dt = \frac{1}{p^2 - 1}. \tag{2.3}$$

- $-1 < \psi(t) < 1, \quad t \in \mathbb{R}.$
- $1 - \psi \in \mathcal{L}_q(\mathbb{R}), \quad 1 \leq q \leq \infty.$
- $\psi'(\pm\infty) = 0; \quad \psi' \in \mathcal{L}_1(\mathbb{R}).$

- $\int_{-\infty}^{\infty}[1-\psi(t)]dt = \int_{-\infty}^{\infty}[1-\psi^{p^2}(t)]dt.$
- $|\int_{-\infty}^{t}[\psi(\tau)-\psi^{p^2}(\tau)]d\tau| < \frac{1}{2\sqrt{\pi}}, \quad t\in\mathbb{R}.$

Denote by $t_k, \quad \ldots < t_m < t_{m-1} < \ldots < t_1$ real zeros and $\sigma_k, k=1,2,\ldots,m$ their multiplicity of function $\psi^{p^2}(t)$. Here $2\le m<\infty$, $m\ge \max_k \sigma_k$; for even p $\sigma_k\ge 2$ is even; for $m=2$ $\sigma_1=\sigma_2=1$, if p is odd, and $\sigma_1=\sigma_2=2$, if p is even.

- $\psi^{p^2}(t) = \frac{a_k}{\sigma_k!}(t-t_k)^{\sigma_k}(1+O(|t-t_k|), \quad t\to t_k$, and the following equalities take place

$$\frac{2^n}{\sqrt{\pi}}\int_{-\infty}^{\infty}(t-\tau)^n e^{-(t-\tau_k)^2}\psi(\tau)d\tau = \begin{cases} a_k\neq 0, \quad n=\sigma_k,\\ 0, n=1,2,\ldots,\sigma_k-1.\end{cases}$$

- $\psi\in\mathcal{L}ip_\alpha(\mathbb{R}), \quad \alpha\ge 1/p^2.$
- $\psi'\in\mathcal{L}_2(\mathbb{R}) \quad$ iff $\sigma_k > p^2/2, k=1,2,\ldots,m.$
- $\int_{-\infty}^{\infty}\psi^2(t)[1-\psi^{p^2-1}(t)]dt \le \frac{1}{4}\|\psi'\|^2.$
- *Sobolev's inequality:* $\quad 4 < \|1-\psi\|\|\psi'\|.$
- *Branching of zeros.* Let $\psi^{p^2}(t)$ has a zero of multiplicity $2n$ at $t=0$. Then the interpolating function $u(1/4-\varepsilon,t)=0$ has pricisely $2n$ simple real roots

$$t_k^{\pm}(\varepsilon) = \pm\lambda_k\sqrt{\varepsilon}+O(\varepsilon) \quad \text{as } \varepsilon\to+0, \quad k=1,2,\ldots,n, \tag{2.4}$$

where $\pm\lambda_k$ are the roots of the Hermite polynomial H_{2n}.

3. Open string

The equation of motion (1.5) is

$$\varphi^p(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\Big[-\frac{(t-\tau)^2}{2}\Big]\varphi(\tau)d\tau, \quad t\in\mathbb{R} \tag{3.1}$$

with boundary conditions (1.8).

For odd p the existence of continuous odd monotonic for $t>0$ solution has been proved (Volovich-jr., 2003). The basic properties of solutions are similar to the closed string ones. However there are some differences, in particular:

For odd p

- $1-|\varphi|, \varphi-\operatorname{sgn} t\in\mathcal{L}_q(\mathbb{R}), \quad 1\le q\le\infty.$
- $\varphi'(\pm\infty)=0, \quad \varphi', |\varphi|'\in\mathcal{L}_1(\mathbb{R}).$
- $\varphi', |\varphi|'\in\mathcal{L}_2(\mathbb{R}) \quad$ iff $\sigma_k > p/2, k=1,2,\ldots,m.$
- $(|\varphi|, |\varphi|-e^{1/2\partial_t^2}|\varphi|) \le \frac{1}{2}\|\varphi'\|^2.$
- *Sobolev's inequality:*
  $8 < \|1-|\varphi|\|^2+\|\varphi'\|^2.$

For even p

- $1-\varphi \in \mathcal{L}_q(a,\infty), \quad 1 \le q \le \infty, \quad a > -\infty.$
- $\varphi' \in \mathcal{L}_1(a,\infty), \quad a > -\infty.$
- $\varphi \in \mathcal{L}ip_\alpha[a,\infty), \quad \alpha \ge \frac{2}{p}, \quad a > -\infty.$
- $\varphi', |\varphi|' \in \mathcal{L}_2(a,\infty)$ iff $\sigma_k > 2/p, t_k \ge a.$

4. Hermite-series expansion of solutions

Definition. Denote by $\mathcal{L}_2^\alpha, \alpha > 0$ the scale of weighted separable Hilbert spaces, consisting of measurable functions square summable on $\mathbb{R}$ with respect to the measure

$$d\mu_\alpha(t) = \sqrt{\frac{\alpha}{\pi}} e^{-\alpha t^2} dt, \quad \int_{-\infty}^{\infty} d\mu_\alpha(t) = 1,$$

with the inner product and norm

$$(f,g)_\alpha = \int_{-\infty}^{\infty} f(t)\bar{g}(t) d\mu_\alpha(t), \quad \|f\| = \sqrt{(f,f)}, \quad f,g \in \mathcal{L}_2^\alpha.$$

- The operator $e^{x\partial_t^2}$ maps continuously $\mathcal{L}_2^\alpha$ into $\mathcal{L}_2^\beta$ for

$$0 < \alpha < \frac{2}{x}, \quad \beta > \frac{\alpha}{1-2\alpha x}.$$

- If $\varphi \in \mathcal{L}_2^1$, then it expands in a series

$$\varphi(t) = \sum_{n=0}^{\infty} a_n \frac{H_n(t)}{2^n n!}, \quad a_n = (\varphi, H_n),$$

which converge in $\mathcal{L}_2^1$; if furthermoore φ is a solution to Eq.(1.9) for $x_0 = 1/4$ then $\varphi^p(t)$ expands in the Taylor series

$$\varphi^p(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!},$$

which converges uniformly on every compact set in $\mathbb{R}$. If the solution belongs to $\mathcal{L}_2^{1/2}$, then the relations

$$(\varphi^p, H_n)_1 = (\varphi, V_n)_{1/2}, \quad n = 0, 1, \ldots$$

hold, where V_n are modified Hermite's polynomials,

$$V_n(t) = 2^{-n/2} H_n(\frac{t}{\sqrt{2}}), \quad n = 0, 1, \ldots.$$

These formulas are used as basic to approximate solutions to b.-v.ps posed (Vladimirov,2005).

5. Tchebyshev-series expantion of periodic solutions

We start with general equation (1.9)

$$\varphi^p(t) = \frac{1}{\sqrt{4\pi x_0}} \int_{-\infty}^{\infty} \exp\Big[-\frac{(t-\tau)^2}{4x_0}\Big]\varphi(\tau)d\tau, \quad t \in \mathbb{R}, \tag{5.1}$$

which is equivalent to the b.-v.p. for the non-linear heat equation

$$\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial t^2}, \quad 0 < x \leq x_0, t \in \mathbb{R}, \tag{5.2}$$

$$u(0,t) = \varphi(t), \quad u(x_0,t) = \varphi^p(t), \quad t \in \mathbb{R}. \tag{5.3}$$

Here, using this method, we constract 2π-periodic solutions to Eq.(5.1). The real 2π-periodic solutions to Eq.(5.2) is

$$u(x,t) = a_0 + \sum_{k=1}^{\infty} e^{-k^2 x}(a_k \cos kt + b_k \sin kt), \tag{5.4}$$

where a_k and b_k are arbitrary real numbers satisfying for instance the condition

$$|a_0| + \sum_{k=1}^{\infty}(|a_k| + |b_k|) < \infty. \tag{5.5}$$

From (5.4) and (5.2) it follows

$$\varphi(t) = a_0 + \sum_{k=1}^{\infty}\Big[a_k T_k(\cos t) + b_k \sin t U_{k-1}(\cos t)\Big], \tag{1}$$

$$\varphi^p(t) = a_0 + \sum_{k=1}^{\infty} e^{-k^2 x_0}\Big[a_k T_k(\cos t) + b_k \sin t U_{k-1}(\cos t)\Big], \tag{5.6}$$

where T_k and U_k are Tchebyshev's polynomials of the first and second genus resp.

How to determine the coefficients a_k and b_k? For pure even or odd solutions we have the following

Theorem 3. *In order the formulas (5.6) represents real 2π- periodic solutions φ to Eq.(5.1), it is necessary and sufficient, that the following equations are fulfiled for all y, $|y| < 1$: for even solutions*

$$\sum_{k=0}^{\infty} e^{-k^2 x_0} a_k T_k(y) = \Big[\sum_{k=0}^{\infty} a_k T_k(y)\Big]^p; \tag{5.7}$$

for odd solutions, p is odd

$$\sum_{k=1}^{\infty} e^{-k^2 x_0} U_{k-1} y) = (1-y^2)^{(p-1)/2}\Big[\sum_{k=1}^{\infty} b_k U_{k-1}(y)\Big]^p. \tag{2}$$

That is the basis fo approximate solutions to Eq.(5.1).
The method can be extanded to many-dimentional case.

6. Applications of the Gauss quadrature formulas

The simplest Gauss qudrature formula is

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-t^2}\varphi(t)dt \approx \sum_{k=1}^{n} \lambda_{kn}\varphi(t_{kn}), \tag{6.1}$$

where points of interpolation t_{kn} are the roots of Hermite's polynomial $H_n(t)$ and coefficients λ_{kn} satisfy the relations

$$\sum_{k=1}^{n} \lambda_{kn} t_{kn}^{2m} = \int_{-\infty}^{\infty} e^{-t^2} t^{2m} dt = \Gamma(m+1/2), m = 0, 1, \ldots, n-1.$$

For instance, for $n = 3$, $\lambda_0 = 2/3$, $\lambda_\pm = 1/6$. $t_0 = 0$, $t_\pm = \pm\sqrt{3/2}$. Applying formula (6.1) to Eq.(5.1) for $x_0 = 1/4$,

$$\varphi^p(t) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\tau^2}\varphi(t-\tau)d\tau,$$

we get approximate Eq.

$$\varphi^p(t) = 2/3\varphi(0) + 1/6\varphi(t-\sqrt{3/2}) + 1/6\varphi(t+\sqrt{3/2}). \tag{6.2}$$

The Eq.(6.2) gives satisfactory approximation to the solution, at least qualitativelly. If we compare the asymptotic behavior at $t \to \infty$ for solution $\varphi(t), \varphi(\infty) = 1$ to Eqs (5.1) and (6.2) for $p = 3$, so one get resp.

$$1 - \varphi(t) \sim \begin{cases} Ce^{-2\sqrt{\log 3}t} = Ce^{-2{,}096t}, \\ Ce^{-\sqrt{3/2}\log 7t} = Ce^{-2{,}384t}. \end{cases}$$

I.Ya.Aref'eva, in her deep astro-physical investigations, made many calculations on computer for more general b.-v.p. ($q^2 = 0.9536$)

$$\varphi^3(t) = \frac{1}{\sqrt{\pi}}(1 - q^2\partial_t^2)\int_{-\infty}^{\infty} e^{-(t-\tau)^2}\varphi(\tau)d\tau, \quad \varphi(\pm\infty) = \pm 1 \tag{6.3}$$

with $n = 3, 5, 7, 9$ points of approximation, and obtained satisfactory approximation to the solution. The method may be used for more general equations of motion in non-flate spaces.

If we represent Eq.(6.3) in the equivalent form

$$\frac{1}{2}\int_0^{\infty} e^{-\tau}[\varphi^3(t-q\tau) + \varphi^3(t+q\tau)]d\tau = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\tau^2}\varphi(t-\tau)d\tau, \tag{6.4}$$

then we may apply to the left hand-side of Eq.(6.4) the quadrature formula with Laguerre's points of interpolation whereas to the right hand-side – the formula (6.1). One may use more precise formulas with weights $e^{-t^2}t^2$, $e^{-t^2}t^{1/3}$,...

Note, that the existence of solutions to b.-v.p. (6.3) was proved by Prokhorenko, 2006, for $0 \le q < 1$; for $q = 0$ it was proved by Volovich-jr, 2004.