# Critical exponent $\nu$ in hierarchical models

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## Correspondence between Euclidean and $p$-adic models

Consider $O(N)$-invariant $N$-component $\varphi^4$ model.

For Euclidean space we have:

$$\nu^{-1} = d/2 + \varepsilon - 2\frac{N+2}{N+8}\varepsilon + \\ + 8\varepsilon^2 \frac{(N+2)(7N+20)}{(N+8)^3} A(d) + O(\varepsilon^3), \tag{1}$$

$$A(d) = -\frac{1}{2}\left(-\gamma - 2\psi(d/4) + \psi(d/2)\right),$$

$$\psi(x) = \Gamma'(x)/\Gamma(x),$$

$$\gamma = \lim_{x \to 0}\left(\frac{1}{x} - \frac{\Gamma(x)}{\text{Res } \Gamma(0)}\right) = 0.577216...$$

$$\varepsilon = \alpha - 3/2.$$

For the $p$-adic space we obtain the same result as in (1), but $\Gamma(x)$ must be replaced by $f_p(x)$:

$$f_p(x) = \left(1 - p^{-2x}\right)^{-1}.$$

$$\nu^{-1} = d/2 + \varepsilon - 2\frac{N+2}{N+8}\varepsilon + \\ + 8\varepsilon^2 \frac{(N+2)(7N+20)}{(N+8)^3} A_p(d) + O(\varepsilon^3), \tag{2}$$

$$A_p(d) = -\frac{1}{2}\left(-\gamma_p - 2\psi_p(d/4) + \psi_p(d/2)\right),$$

$$\psi_p(x) = f_p'(x)/f_p(x),$$

$$\gamma_p = \lim_{x\to 0}\left(\frac{1}{x} - \frac{f_p(x)}{\text{Res } f_p(0)}\right) = -\ln p.$$

**Main results**

- Critical exponent $\nu$ in $N$-component Fermionic hierarchical model may be calculated via finding roots of some well defined finite polynomial.

- In a case of 4-component model an explicit expression for $\nu$ was found. It is in full agreement with the result (2).

## $N$-component hierarchical model

Let's define $2N$-component fermionic field:

$$\psi(j) = (\psi_1^+(j), \psi_1^-(j), \dots, \psi_N^+(j), \psi_N^-(j)),$$

where $\psi_k^+(j), \psi_k^-(j)$, are generators of Grassmann algebra, $j \in \mathbb{Z}$, $k = 1, \dots, N$.

Distribution function of the field is formally defined as

$$F(\psi) = \exp\left(-H_0(\psi)\right) \prod_{j \in \mathbb{Z}} f(\psi(j)).$$

The Gaussian part of the hamiltonian $H_0(\psi)$ is invariant under renormalization group (RG) transformation with parameter $\alpha$, and self-interaction is given by:

$$f(\psi) = c_0 + \sum_{k=1}^{N} \frac{c_k}{k!} \left( \sum_{i=1}^{N} \psi_i^+ \psi_i^- \right)^k.$$

Spin-block renormalization group transformation is defined by the equation:

$$\psi'(i) = n^{-\alpha/2} \sum_{j \in V_i} \psi(j), \tag{3}$$

$\alpha \in \mathbb{R}$ is a parameter of renormalization group,

$$V_i = \{j \in \mathbb{Z} : (i-1)n < j \leq in\}.$$

RG transformation may be represented in the integral form:

$$\begin{aligned} f'(\psi) &= \exp(n^{\alpha-1}\psi^+\psi^-)\times \\ &\times \int d\xi \exp(-i\, n^{\alpha/2}\xi\psi)\times \\ &\quad \times \left[\int \exp(-\phi^+\phi^- + i\,\xi\phi) f(\phi)\, d\phi\right]^n. \end{aligned} \tag{4}$$

Let's denote

$$w_j = \left(1 - n^{\alpha-1}\right)^j \sum_{s=0}^{N-j} \binom{N-j}{s} c_s (-1)^{N-j-s},$$

and $v_k = w_k/w_0$. In a space of coefficients $v_1, \ldots, v_N$ the RG-transformation is a rational transformation:

$$v'_t = \frac{\sum_{j=t}^{N} (-1)^{j-t} n^{-j\alpha} \binom{N-t}{N-j} T_j(v_1, \ldots, v_N)}{\sum_{j=0}^{N} (-1)^{j} n^{-j\alpha} \binom{N}{N-j} T_j(v_1, \ldots, v_N)}, \tag{5}$$

where

$$T_j(v_1, \ldots, v_N) = \\ = \sum_{t=1}^{j} \sum_{\substack{l_1 + 2\,l_2 + \cdots + j\,l_j = j \\ l_1 + \cdots + l_j = t}} \frac{n^{[t]}}{l_1! \ldots l_j!} \times \\ \times \frac{j!}{1!^{l_1} 2!^{l_2} \ldots j!^{l_j}} \prod_{i=1}^{j} v_i^{l_i}, \tag{6}$$

$$n^{[t]} = n(n-1)\ldots(n-t+1).$$

RG-transformation (5) has two trivial fixed points:

- $v_j = 0$, $j = 1, \ldots, N$, corresponding to Grassmann $\delta$-function.
- Gaussian fixed point $v_j = \left(n^{\alpha-1} - 1\right)^j$, $j = 1, \ldots, N$.

## 4-component model

For $N = 2$ RG-transformation (5) has a form

$$v_1' = \frac{anv_1 - v_2 - (n-1)v_1^2}{a^2 n - 2anv_1 + v_2 + (n-1)v_1^2}, \qquad (7)$$

$$v_2' = \frac{v_2 + (n-1)v_1^2}{a^2 n - 2anv_1 + v_2 + (n-1)v_1^2}, \qquad (8)$$

where

$$a = n^{\alpha - 1}.$$

This transformation has 4 fixed points: two above mentioned trivial points and two non-trivial points.

$$v_1^{(1)} = 0, \quad v_2^{(1)} = 0$$

$$v_1^{(2)} = a - 1, \quad v_2^{(2)} = (a-1)^2$$

$$v_1^{(3)} = \frac{a\sqrt{n} - 1}{\sqrt{n} + 1}, \quad v_2^{(3)} = \frac{(\sqrt{n} - 1)(a\sqrt{n} - 1)}{\sqrt{n} + 1},$$

$$v_1^{(4)} = \frac{a\sqrt{n} + 1}{\sqrt{n} - 1}, \quad v_2^{(4)} = -\frac{(\sqrt{n} + 1)(a\sqrt{n} + 1)}{\sqrt{n} - 1},$$

We are interested in calculation of critical exponent $\nu$. Let $n = p^d$, $d$ is a dimension of space.

$$\nu = \frac{\ln p}{\ln x},$$

where $x$ is the greatest eigenvalue of differential $\mathcal{D}R$ of RG-transformation (7)-(8) at the non-Gaussian fixed point.

$$\det ||\mathcal{D}R|_v - xI|| = 0, \quad v = Rv.$$

For 4-component Fermionic model:

$$x^2(n^{-1}-1)+x\left(4a+6+4a^{-1}+2n^{-1}\right)+ \\ +n^{-1}+n-4a^2-8a-10-8a^{-1}-4a^{-2}+ \\ +x^{-1}\left(4a+6+4a^{-1}+2n\right)+(n-1)x^{-2}=0. \tag{9}$$

The variable $a$ is related to critical exponent $\eta$ via the identity

$$a = p^{2-\eta}.$$

For general $N$ we show that $x$ may be calculated as a root of some finite polynomial.

All solutions of equation (9) are calculated explicitly. One of them gives the following expression for $\nu$:

$$\nu^{-1} = d/2 + 2\varepsilon d - \frac{d(1+4\sqrt{n}+n)\varepsilon^2 \ln(n)}{n-1} + \\ + \frac{d(1+10\sqrt{n}+n)\varepsilon^3 \ln(n)^2}{(\sqrt{n}-1)^2} + O(\varepsilon^4), \tag{10}$$

where $\varepsilon = \alpha - 3/2$.

This expression was compared with the expression, earlier obtained for $p$-adic $\varphi^4$-model using methods of quantum field theory in terms of Feynmann diagrams. Both results are in full correspondence with each other.

It is interesting to discover analogous property for $\nu$ for Euclidean $\varphi^4$ model.