p-adic Brownian Motion over $\mathbb{Q}_p$ *

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Earlier Research on p-adic Brownian Motion

- A. Kh. Bikulov and I.V. Volovich, *p-adic Brownian Motion*, Izv. Math. 61 (1997) pp. 537–552.

- A. Kh. Bikulov, *Stochastic p-adic Equations of Mathematical Physics*, Theoret. and Math. Phys. 119 (1999), pp. 594–604.

- A. Yu. Khrennikov and S.V. Kozyrev, *Ultrametric Random Field*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 9 (2006) pp. 199–213.

- K. Kamizono, *Symmetric Stochastic Integrals with respect to p-adic Brownian Motion*, To Appear in Stochastics.

Plan of This Talk

- Present an explicit construction of p-adic Brownian motion over $\mathbb{Q}_p$ using the Paley-Wiener method.

- Introduce p-adic random walk over $\mathbb{Q}_p/\mathbb{Z}_p$.

- Discuss the weak convergence of p-adic random walk over to p-adic Brownian motion (unfinished).

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Notation

Write

$$I_n \triangleq \left\{t = t_n p^{-n} + \ldots + t_j p^{-1} \mid t_j \in \{0, 1, \ldots, p-1\}\right\}, \quad I \triangleq \bigcup_{n \in \mathbb{N}} I_n \simeq \mathbb{Q}_p/\mathbb{Z}_p.$$

Every $t \in \mathbb{Q}_p$ can be uniquely decomposed as

$$t = [t] + \{t\}, \quad [t] \in \mathbb{Z}_p : \textit{integer part}, \quad \{t\} \in I : \textit{fractional part}.$$

Denote by χ the normalized additive character of $\mathbb{Q}_p$:

$$\chi(t) \triangleq e^{2\pi\sqrt{-1}\{t\}}, \quad t \in \mathbb{Q}_p.$$

Let $\mathbb{L}^2(\mathbb{Q}_p)$ be the space of square-integrable, complex-valued functions over $\mathbb{Q}_p$ with respect to the normalized Haar measure. Let $\mathscr{D}(\mathbb{Q}_p)$ be the space of Schwartz-Bruhat functions over $\mathbb{Q}_p$ and let $\mathscr{D}'(\mathbb{Q}_p)$ be its topological dual. Denote by $\hat{u}$ the Fourier transform of u:

$$\hat{\varphi}(k) \triangleq \int_{\mathbb{Q}_p} \varphi(t)\chi(kt)dt, \quad k \in \mathbb{Q}_p, \quad \text{for} \quad \varphi \in \mathscr{D}(\mathbb{Q}_p)$$

$$\langle \hat{f}, \varphi \rangle \triangleq \langle f, \hat{\varphi} \rangle, \quad \varphi \in \mathscr{D}(\mathbb{Q}_p), \quad \text{for} \quad f \in \mathscr{D}'(\mathbb{Q}_p).$$

Denote by $\mathbf{D}$ the first order Vladimirov differential operator:

$$\mathbf{D}f \triangleq f_{-1} * f, \quad \text{for} \quad f \in \mathscr{D}'(\mathbb{Q}_p),$$

if the right-hand-side exists, where $*$ is the convolution operator and

$$\langle f_{-1}, \varphi \rangle \triangleq -\frac{p^2}{p+1}\Big[\int_{\mathbb{Z}_p} \frac{\varphi(t)-\varphi(0)}{|t|_p^2}dt + \int_{\mathbb{Q}_p\setminus\mathbb{Z}_p} \frac{\varphi(t)}{|t|_p^2}dt - \frac{1}{p}\varphi(0)\Big], \quad \varphi \in \mathscr{D}(\mathbb{Q}_p).$$

For any subset $A \subseteq \mathbb{Q}_p$, denote by $\mathbf{1}_A$ the indicator function of A:

$$\mathbf{1}_A(t) \triangleq \left\{ \begin{array}{lll} 1 & \text{for} & t \in A \\ 0 & \text{for} & t \in \mathbb{Q}_p\setminus A \end{array} \right\}.$$

p-adic White Noise over $\mathbb{Q}_p$

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. All random variables will be complex-valued and defined on $(\Omega, \mathscr{F}, \mathbb{P})$ unless stated otherwise.

Definition 1 (p-adic White Noise)**.** A *p-adic white noise over* $\mathbb{Q}_p$ is a random variable X taking values in $\mathscr{D}'(\mathbb{Q}_p)$ such that the family $\{\langle X, \varphi\rangle\}_{\varphi \in \mathscr{D}(\mathbb{Q}_p)}$ is Gaussian with mean 0 and covariance

(1) $$\mathbb{E}[\langle X, \varphi\rangle \overline{\langle X, \psi\rangle}] = \int_{\mathbb{Q}_p} \varphi(t)\overline{\psi(t)}dt \quad \forall \varphi, \psi \in \mathscr{D}(\mathbb{Q}_p).$$

Introduce the family of functions $\{\psi_{k,l}\}_{k,l\in I} \subseteq \mathscr{D}(\mathbb{Q}_p)$ given by

(2) $$\psi_{k,l}(t) \triangleq \mathbf{1}_{k+\mathbb{Z}_p}(t)\chi(l(t-k)), \quad t \in \mathbb{Q}_p.$$

Lemma 2. *The family* $\{\psi_{k,l}\}_{k,l\in I}$ *forms a complete orthonormal system of* $\mathbb{L}^2(\mathbb{Q}_p)$.

Proof. It is easy to check that $\{\psi_{k,l}\}_{k,l\in I}$ forms an orthonormal system of $\mathbb{L}^2(\mathbb{Q}_p)$. Let us prove completeness. Let $\varphi \in \mathscr{D}(\mathbb{Q}_p)$. Take a large $m \in \mathbb{N}$ such that $\operatorname{supp}(\varphi) \cup \operatorname{supp}(\hat{\varphi}) \subseteq p^{-m}\mathbb{Z}_p$. From the Parseval-Steklov equation, we have

$$\|\varphi\|^2_{\mathbb{L}^2(\mathbb{Q}_p)} = \|\hat{\varphi}\|^2_{\mathbb{L}^2(\mathbb{Q}_p)} = \sum_{l\in I_m} \int_{l+\mathbb{Z}_p} \Big| \sum_{k\in I_m} \int_{k+\mathbb{Z}_p} \varphi(t)\chi(ut)dt \Big|^2 du. \tag{3}$$

Now, if $u \in l + \mathbb{Z}_p$ and $t \in k + \mathbb{Z}_p$, we have $\chi((u-l)(t-k)) = 1$. Thus, for $u \in l + \mathbb{Z}_p$,

$$\begin{aligned}\Big| \sum_{k\in I_m} \int_{k+\mathbb{Z}_p} \varphi(t)\chi(ut)dt \Big|^2 &= \Big| \sum_{k\in I_m} \chi(uk) \int_{k+\mathbb{Z}_p} \varphi(t)\chi(l(t-k))dt \Big|^2 \\ &= \sum_{k_1,k_2\in I_m} \chi(u(k_1-k_2))(\varphi, \psi_{k_1,-l})_{\mathbb{L}^2(\mathbb{Q}_p)} \overline{(\varphi, \psi_{k_2,-l})_{\mathbb{L}^2(\mathbb{Q}_p)}}.\end{aligned} \tag{4}$$

It follows from (3) and (4) that

$$\|\varphi\|^2_{\mathbb{L}^2(\mathbb{Q}_p)} = \sum_{k_1,k_2,l\in I_m} (\varphi, \psi_{k_1,-l})_{\mathbb{L}^2(\mathbb{Q}_p)} \overline{(\varphi, \psi_{k_2,-l})_{\mathbb{L}^2(\mathbb{Q}_p)}} \int_{l+\mathbb{Z}_p} \chi(u(k_1 - k_2))du$$
$$= \sum_{k,l\in I_m} (\varphi, \psi_{k,-l})_{\mathbb{L}^2(\mathbb{Q}_p)} \overline{(\varphi, \psi_{k,-l})_{\mathbb{L}^2(\mathbb{Q}_p)}}.$$

Finally, let $m \to \infty$ to obtain the result. □

Proposition 3. *The probability distribution of p-adic white noise X is not supported on the space $\mathscr{E}'(\mathbb{Q}_p)$ of Schwartz distributions with compact support.*

Proof. Suppose that $X \in \mathscr{E}'(\mathbb{Q}_p)$ a.s. Then, for a.e. $\omega \in \Omega$, there exists $n_\omega \in \mathbb{N}$ such that $\operatorname{supp} X(\omega) \subseteq p^{-n_\omega}\mathbb{Z}_p$. Now, since $\operatorname{supp} \psi_{k,l} \subseteq k+\mathbb{Z}_p$, we have $\langle X(\omega), \psi_{k,l}\rangle = 0$ for $k \in I \setminus I_{n_\omega}$. This implies that $\langle X, \psi_{k,l}\rangle \to 0$ as $|k|_p \to \infty$ a.s., and therefore,

$$e^{-\theta^2/4} = \mathbb{E}\left[e^{\sqrt{-1}\,\theta\,\mathrm{Re}\,\langle X,\psi_{k,l}\rangle}\right] \to 1 \quad (|k|_p \to \infty) \quad \forall\theta \in \mathbb{R},$$

a contradiction. □

p-adic Brownian Motion over $\mathbb{Q}_p$

Definition 4 (p-adic Brownian Motion over $\mathbb{Q}_p$)**.** A *p-adic Brownian motion over* $\mathbb{Q}_p$ is a stochastic process $\{W(t)\}_{t\in\mathbb{Q}_p}$ such that

(i) the family $\{W(t)\}_{t\in\mathbb{Q}_p}$ is Gaussian with mean 0 and covariance

$$\mathbb{E}\big[W(t)\overline{W(s)}\big] = \frac{p+1}{p^2}\big(|t|_p + |s|_p - |t-s|_p\big) \quad \forall t, s \in \mathbb{Q}_p; \tag{5}$$

(ii) the sample path $t \mapsto W(t,\omega)$ is continuous for every fixed $\omega \in \Omega$.

Remark 5. Similar properties to the classical Brownian motion with real time index:

(i) $\mathbb{E}[W(t)\overline{W(s)}] = \dfrac{p+1}{p^2}\min(|t|_p, |s|_p)$ if $|t|_p \neq |s|_p$.

(ii) $W(t_2) - W(t_1)$ and $W(t_4) - W(t_3)$ are mutually independent if $0 \leq |t_1|_p \leq |t_2|_p < |t_3|_p \leq |t_4|_p$.

Construction of p-adic Brownian Motion over $\mathbb{Q}_p$

Consider the stochastic equation:

$$\mathbf{D}\Phi = X(\omega), \quad \Phi(0) = 0, \tag{6}$$

where X is p-adic white noise over $\mathbb{Q}_p$. If $X(\omega) \in \mathscr{E}'(\mathbb{Q}_p)$, then (6) would have a unique solution. However, as we have seen, $\mathbb{P}(\omega : X(\omega) \in \mathscr{E}'(\mathbb{Q}_p)) \neq 1$. Thus, (6) does not have a solution in almost sure sense. We will take the Paley-Wiener method to construct p-adic Brownian Motion over $\mathbb{Q}_p$. This is a direct generalization of [A. Kh. Bikulov and I.V. Volovich, Izv. Math. 61 (1997) pp. 537–552].

Suppose first that there exists on $(\Omega, \mathscr{F}, \mathbb{P})$ an independent family $\{Z_{k,l}\}_{k,l\in I}$ of standard Gaussian random variables. For each $\varphi \in \mathscr{D}(\mathbb{Q}_p)$, let

$$\langle X(\omega), \varphi \rangle \triangleq \sum_{k,l\in I} Z_{k,l}(\omega) \langle \psi_{k,-l}, \varphi \rangle \quad \omega \in \Omega. \tag{7}$$

The family $\{X_\varphi\}_{\varphi\in\mathscr{D}(\mathbb{Q}_p)}$ is clearly p-adic white noise.

Next, consider the following function

$$(8)\quad G(t,s) \triangleq -\frac{1-p^{-1}}{\log p}\Big(\log|t-s|_p - \log|s|_p\Big), \quad (t,s)\in\mathbb{Q}_p\times\mathbb{Q}_p,\ t\neq s,\ s\neq 0.$$

For every fixed $t\in\mathbb{Q}_p$, the function $G(t,\cdot)$ belongs to $\mathbb{L}^1(\mathbb{Q}_p)$ with Fourier transform

$$(9)\qquad \hat{G}(t,\cdot)(k) = \frac{\chi(kt)-1}{|k|_p} \quad \forall k\in\mathbb{Q}_p\setminus\{0\} \quad \text{and} \quad \hat{G}(t,\cdot)(0)=0.$$

Let us consider the integral operator

$$(10)\qquad T\varphi(t) \triangleq \int_{\mathbb{Q}_p} G(t,s)\varphi(s)ds, \quad t\in\mathbb{Q}_p.$$

For every $f\in\mathscr{D}(\mathbb{Q}_p)$, $u=Tf$ is the unique solution to

$$(11)\qquad \mathbf{D}u = f \ \text{ on } \mathbb{Q}_p, \quad u(0)=0.$$

We would like to solve, in some sense, the stochastic equation:

$$\mathbf{D}W = X, \quad W(0) = 0,$$

where X is p-adic white noise:

$$X = \sum_{k,l \in I} Z_{k,l} \psi_{k,-l}.$$

If X were random variable taking values in $\mathscr{D}(\mathbb{Q}_p)$, the solution could be written as

$$W = TX = \sum_{k,l \in I} Z_{k,l} T\psi_{k,-l}.$$

With this observation in mind, it is natural to take the limit

(12) $$W(t) \triangleq \lim_{n \to \infty} \sum_{k,l \in I_n} Z_{k,l} T\psi_{k,-l}(t), \quad t \in \mathbb{Q}_p$$

as a candidate of p-adic Brownian motion over $\mathbb{Q}_p$.

Theorem 6 (Existence of p-adic Brownian Motion over $\mathbb{Q}_p$). *Let* $(\Omega, \mathscr{F}, \mathbb{P})$ *be a probability space on which there exists an independent family* $\{Z_{k,l}\}_{k,l\in I}$ *of standard Gaussian random variables. Let* $\{\psi_{k,l}\}_{k,l\in I}$ *be the complete orthonormal system of* $\mathbb{L}^2(\mathbb{Q}_p)$ *given by* (2), *and let* T *be the integral operator given by* (10) *with integral kernel* (8). *Then,*

(i) *the limit in* (12) *exists in* $\mathbb{L}^2(\Omega)$ *for every fixed* $t \in \mathbb{Q}_p$;

(ii) *the family* $\{W(t)\}_{t\in\mathbb{Q}_p}$ *is Gaussian with mean* 0 *and covariance*

$$\mathbb{E}\big[W(t)\overline{W(s)}\big] = \frac{p+1}{p^2}\big(|t|_p + |s|_p - |t-s|_p\big).$$

(iii) *there exists a continuous modification of* $\{W(t)\}_{t\in\mathbb{Q}_p}$.

Proof. First, let us calculate $T\psi_{k,-l}$. For $l = 0$, we have $\psi_{k,0}(s) = \mathbf{1}_{k+\mathbb{Z}_p}(s)$ and thus, $T\psi_{k,0}(t) = G(t,s) * \mathbf{1}_{\mathbb{Z}_p}(k)$. Let $l \neq 0$. In this case,

$$(13) \qquad T\psi_{k,-l}(t) = \chi(lk)\int_{\mathbb{Q}_p} \mathbf{1}_{k+\mathbb{Z}_p}(s)G(t,s)\chi(-ls)ds$$
$$= \chi(lk)(\mathbf{1}_{k+\mathbb{Z}_p}\cdot G(t,\cdot))\hat{}(-l) = \chi(lk)(\hat{\mathbf{1}}_{k+\mathbb{Z}_p} * \hat{G}(t,\cdot))(-l).$$

Now, since $\hat{\mathbf{1}}_{k+\mathbb{Z}_p} = \int_{k+\mathbb{Z}_p} \chi(us)ds = \chi(uk)\mathbf{1}_{\mathbb{Z}_p}(u)$, it follows from (9) that

$$(\hat{\mathbf{1}}_{k+\mathbb{Z}_p} * \hat{G}(t,\cdot))(-l) = \int_{\mathbb{Q}_p} \mathbf{1}_{\mathbb{Z}_p}(l+u)\chi(-(l+u)k)\frac{\chi(ut)-1}{|u|_p}du$$
$$= \int_{\mathbb{Z}_p} \chi(-vk)\frac{\chi((v-l)t)-1}{|v-l|_p}dv$$
$$= \left\{ \begin{array}{ll} \dfrac{\chi(-lt)\mathbf{1}_{\mathbb{Z}_p}(t)-1}{|l|_p} & \text{if } k=0 \\ \dfrac{\chi(-lt)\mathbf{1}_{k+\mathbb{Z}_p}(t)}{|l|_p} & \text{if } k\neq 0 \end{array} \right\}. \tag{14}$$

Combining (13) and (14), we obtain

$$T\psi_{k,-l}(t) = \left\{ \begin{array}{ll} \dfrac{\chi(-l[t])\mathbf{1}_{\mathbb{Z}_p}(t)-1}{|l|_p} & \text{if } k=0 \\ \dfrac{\chi(-l[t])\mathbf{1}_{k+\mathbb{Z}_p}(t)}{|l|_p} & \text{if } k\neq 0 \end{array} \right\}. \tag{15}$$

The series (12) then becomes

$$(16) \qquad W(t) = \lim_{n\to\infty}\Big[\sum_{k\in I_n} Z_{k,0}(G(t,\cdot) * \mathbf{1}_{\mathbb{Z}_p})(k) + \sum_{l\in I_n\setminus\{0\}} \frac{Z_{\{t\},l}\chi(-l[t]) - Z_{0,l}}{|l|_p}\Big].$$

To prove the convergence in $\mathbb{L}^2(\Omega)$ of the right-hand-side of (16), notice first that the first term of (16) stabilizes for large n because the summand vanishes if $|k|_p > |t|_p$. To see the convergence of the second term, we calculate as

$$\mathbb{E}\Big|\sum_{l\in I_n\setminus\{0\}} \frac{Z_{\{t\},0}\chi(-l[t]) - Z_{0,l}}{|l|_p}\Big|^2$$

$$= \left\{ \begin{array}{ll} \displaystyle\sum_{l\in I_n\setminus\{0\}} \frac{(\chi(-lt)-1)(\chi(lt)-1)}{|l|_p^2} \to \frac{2(p+1)}{p^2}|t|_p \quad (n\to\infty) & \text{if } t\in\mathbb{Z}_p \\ \displaystyle\sum_{l\in I_n\setminus\{0\}} \frac{\chi(-l[t])\chi(l[t])}{|l|_p^2} + \sum_{l\in I_n\setminus\{0\}} \frac{1}{|l|_p^2} \to \frac{2}{p} \quad (n\to\infty) & \text{if } t\in\mathbb{Q}_p\setminus\mathbb{Z}_p \end{array} \right\}.$$

This proves **(i)**.

It is obvious that $\{W(t)\}_{t\in\mathbb{Q}_p}$ is Gaussian with mean 0. Let us calculate the covariance. If $t, s \in \mathbb{Z}_p$, the right-hand-side of (16) coincides, except for the constant multiple, with the p-adic Brownian motion over $\mathbb{Z}_p$ of Bikulov and Volovich (1997). Thus, (5) holds for $t, s \in \mathbb{Z}_p$. If $t \in \mathbb{Z}_p$ and $s \in \mathbb{Q}_p \setminus \mathbb{Z}_p$, then

$$\mathbb{E}[W(t)\overline{W(s)}] = -\lim_{l\in I_n\setminus\{0\}} \frac{\chi(lt)-1}{|l|_p^2} = \frac{p+1}{p^2}|t|_p,$$

and the right-hand-side coincides with that of (5). If $t, s \in \mathbb{Q}_p \setminus \mathbb{Z}_p$, write

$$\mathbb{E}[W(t)\overline{W(s)}] = S_1(t,s) + S_2(t,s), \tag{17}$$

where $S_1(t,s)$ and $S_2(t,s)$ are given by

$$S_1(t,s) \triangleq \left\{ \begin{array}{ll} \displaystyle\sum_{l\in I\setminus\{0\}} \frac{\chi(-l([t]-[s]))+1}{|l|_p^2} = \frac{2}{p} - \frac{p+1}{p^2}|t-s|_p & \text{if } |t-s|_p \le 1 \\ \displaystyle\sum_{l\in I\setminus\{0\}} \frac{1}{|l|_p^2} = \frac{1}{p} & \text{if } |t-s|_p > 1 \end{array} \right\},$$

as well as

$$
\begin{aligned}
S_2(t,s) &\triangleq \sum_{k\in I}(G(t,\cdot)*\mathbf{1}_{\mathbb{Z}_p})(k)(G(s,\cdot)*\mathbf{1}_{\mathbb{Z}_p})(k)\\
&= \int_{\mathbb{Q}_p}(G(t,\cdot)*\mathbf{1}_{\mathbb{Z}_p})(k)(G(s,\cdot)*\mathbf{1}_{\mathbb{Z}_p})(k)dk\\
&= \int_{\mathbb{Q}_p}\Big(\int_{\mathbb{Z}_p} G(t,k-u)du\Big)\Big(\int_{\mathbb{Z}_p} G(s,k-v)dv\Big)dk\\
&= \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\Big(\int_{\mathbb{Q}_p} G(t,k-u)G(s,k-v)dk\Big)dudv\\
&= \left\{\begin{array}{ll}
\dfrac{p+1}{p^2}(|t|_p+|s|_p)-\dfrac{2}{p} & \text{if } |t-s|_p\le 1\\
\dfrac{p+1}{p^2}\big(|t|_p+|s|_p-|t-s|_p\big)-\dfrac{1}{p} & \text{if } |t-s|_p> 1
\end{array}\right\}.
\end{aligned}
$$

It follow that (5) is valid for $t,s\in\mathbb{Q}_p\setminus\mathbb{Z}_p$ as well. This proves **(ii)**.

Finally, for each $k \in I$, let

$$W_k(t) \triangleq W(t+k), \quad t \in \mathbb{Z}_p.$$

Then, we see

$$\mathbb{E}\left|W_k(t) - W_k(s)\right|^{2n} = \left(\frac{4(p+1)^4}{p^8}\right)^n n!\left|t-s\right|_p^n \quad \forall t, s \in \mathbb{Z}_p, \forall n \in \mathbb{N},$$

and therefore it follows from Lemma 7 below that there exists for every $k \in I$ a continuous modification $\{\tilde{W}_k(t)\}_{t\in\mathbb{Z}_p}$ for $\{W_k(t)\}_{t\in\mathbb{Z}_p}$. Since the I is countable and $\mathbb{Q}_p$ is the union of disjoint open subsets $k + \mathbb{Z}_p$ with $k \in I$, we conclude that $\{\tilde{W}_{\{t\}}([t])\}_{t\in\mathbb{Q}_p}$ is a continuous modification of $\{W(t)\}_{t\in\mathbb{Q}_p}$ $\square$

Lemma 7 (The Kolmogorov-Čentsov Criterion). *Let $\{X(t)\}_{t\in\mathbb{Z}_p}$ be a stochastic process for which there exist constants $\alpha > 0$, $\beta > 0$ and $C > 0$ such that*

$$\mathbb{E}\,|X(t) - X(s)|^{\alpha} \leq C\,|t-s|_p^{1+\beta} \quad \forall t, s \in \mathbb{Z}_p. \tag{18}$$

Then, $\{X(t)\}_{t\in\mathbb{Z}_p}$ has a continuous modification $\{\tilde{X}(t)\}_{t\in\mathbb{Z}_p}$ which is locally Hölder continuous of exponent γ for every $0 < \gamma < \beta/\alpha$, that is, there exist a constant $\delta > 0$ and a positive random variable h such that

$$\sup_{0<|t-s|_p<h(\omega)} \frac{|\tilde{X}(t,\omega) - \tilde{X}(s,\omega)|}{|t-s|_p^{\gamma}} \leq \delta, \quad \forall \omega \in \Omega. \tag{19}$$

Proof. See Appendix of [K. Kamizono "Symmetric Stochastic Integrals with respect to p-adic Brownian Motion", To Appear in Stochastics]. □

p-adic Difference Equations over $I \simeq \mathbb{Q}_p/\mathbb{Z}_p$

Let us consider a discrete time analogue of the Vladimirov differential operator. Consider the spaces

$$\mathscr{D}' \triangleq \left\{ \varphi = \{\varphi_k\}_{k\in I} \,\middle|\, \varphi_k \in \mathbb{C} \quad \forall k \in I \right\}$$
$$\mathscr{D} \triangleq \left\{ \varphi \in \mathscr{D}' \,\middle|\, \operatorname{supp}\varphi \subseteq I_n \quad \text{for some } n \in \mathbb{N} \right\}.$$

We say that $\varphi \in \mathscr{D}'$ is *differenceable* if the limit

$$\begin{aligned}\Delta\varphi_k &\triangleq -\frac{p^2}{p+1}\lim_{n\to\infty}\sum_{l\in I_n\setminus\{0\}}\frac{\varphi_{k+l}}{|l|_p^2} + \frac{p}{p+1}\varphi_k \\ &= -\frac{p^2}{p+1}\lim_{n\to\infty}\sum_{l\in I_n\setminus\{0\}}\frac{\varphi_{k+l}-\varphi_k}{|l|_p^2}\end{aligned} \tag{20}$$

exists for all $k \in I$. In this case, we call $\Delta\varphi \triangleq \{\Delta\varphi_k\}_{k\in I}$ the *difference* of φ.

Let

$$G_{k,l} \triangleq \left\{ \begin{array}{ll} -\frac{1-p^{-1}}{\log p}\Big(\log|k-l|_p - \log|l|_p\Big) & \text{if } k \neq l,\ l \neq 0 \\ -\frac{1-p^{-1}}{\log p}\Big(\log|k|_p + \frac{p^{-1}\log p}{1-p^{-1}}\Big) & \text{if } k \neq l,\ l = 0 \\ -\frac{1-p^{-1}}{\log p}\Big(-\frac{p^{-1}\log p}{1-p^{-1}} - \log|k|_p\Big) & \text{if } k = l,\ l \neq 0 \\ 0 & \text{if } k = l = 0 \end{array} \right\}. \tag{21}$$

Notice that $G_{k,\cdot} \in \mathscr{D}$ for each $k \in I$ and

$$G\varphi_k \triangleq \sum_{l \in I} G_{k,l}\varphi_l, \quad k \in I$$

is well-defined for $\varphi \in \mathscr{D}'$.

Theorem 8. *If $\varphi \in \mathscr{D}$ and $c \in \mathbb{C}$, then there exists a unique solution to*

$$\Delta X = \varphi, \quad X_0 = c \tag{22}$$

and the solution is written as

$$X = G\varphi + c. \tag{23}$$

p-adic Random Walk over $I \simeq \mathbb{Q}_p/\mathbb{Z}_p$

Definition 9. A *p-adic symmetric random walk* is the family $S = \{S_k\}_{k\in I}$ of random variable given by

(24) $$S_k(\omega) \triangleq GZ_k(\omega) = \sum_{l\in I} G_{k,l} Z_l(\omega), \quad k \in I,\ \omega \in \Omega,$$

where $Z = \{Z_k\}_{k\in I}$ is an independent family of random variables with $\mathbb{E}[Z_k] = 0$, $\mathbb{E}|Z_k|^2 = 1$ and $\mathbb{E}|Z_k|^3 < \infty$ on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

It can be shown that

(i) S is differenceable a.s. with $\Delta S = Z$, and

(ii) S is a centered Gaussian random variable with

(25) $$\begin{aligned} \mathbb{E}[S_k \overline{S_l}] &= \frac{p+1}{p^2}(|k|_p + |l|_p - |k-l|_p) - \frac{1}{p}, \quad \text{if} \quad k \neq l \\ \mathbb{E}|S_k|^2 &= \frac{2(p+1)}{p^2}|k|_p - \frac{2}{p}. \end{aligned}$$

Weak Convergence

Let $S = \{S_k\}_{k \in I}$ be a p-adic random walk. For each $n \in \mathbb{N}$, let

$$W^{(n)}(t) \triangleq p^{-n/2} S_{\{p^{-n} t\}}, \quad t \in \mathbb{Q}_p. \tag{26}$$

Then, $\{W^{(n)}(t)\}_{t \in \mathbb{Q}_p}$ induces a probability measure $\mu^{(n)}$ on the space $(C(\mathbb{Q}_p), \mathscr{B}(C(\mathbb{Q}_p)))$ of continuous functions.

Conjecture. As $n \to \infty$, the sequence $\mu^{(n)}$ converges weakly to a probability measure μ with respect to which the process $\{w(t)\}_{t \in \mathbb{Q}_p}$ given by $C(\mathbb{Q}_p) \ni w \mapsto w(t)$ is a p-adic Brownian motion.

We need to show:

(i) Every finite dimensional distribution of $\{\mu^{(n)}\}_{n \in \mathbb{N}}$ converges weakly to that of μ.

(ii) $\{\mu^{(n)}\}_{n \in \mathbb{N}}$ is tight.

I managed to prove **(i)**, but **(ii)** is still open.

Lemma 10. *For every distinct $t_1, \ldots, t_d \in \mathbb{Q}_p$, the probability distribution of $(W^{(n)}(t_1), \ldots, W^{(n)}(t_d))$ converges weakly to the centered Gaussian with covariance matrix $\Sigma(t_1, \ldots, t_d))$, where $\Sigma(t_1, \ldots, t_d) \triangleq (B(t_i, t_j))$ with*

$$B(t,s) \triangleq \frac{p+1}{p^2}(|t|_p + |s|_p - |t-s|_p), \quad t, s \in \mathbb{Q}_p. \tag{27}$$

Proof. For simplicity, we consider the real-valued case. It suffices to show $\varphi^{(n)}(\theta) \to \exp(-\theta'\Sigma(t_1, \ldots, t_d)\theta/2)$ $(n \to \infty)$ $\forall \theta \in \mathbb{R}^d$, where $\varphi^{(n)}(\theta) \triangleq \mathbb{E}[\exp(\sqrt{-1}\sum_{j=1}^d \theta_j W^{(n)}(t_j))]$. Without loss of generality, we may assume $|t_1|_p \leq \cdots \leq |t_d|_p$. By definition,

$$\sum_{j=1}^d \theta_j W^{(n)}(t_j) = p^{-n/2}\sum_{j=1}^d \theta_j \sum_{l \in I} G_{\{p^{-n}t_j\},l} Z_l = p^{-n/2}\sum_{l \in I} X_{n,l},$$

where

$$X_{n,l} \triangleq \sum_{j=1}^d \theta_j G_{\{p^{-n}t_j\},l} Z_l.$$

For each $n \in \mathbb{N}$, the family $\{X_{n,l}\}_{l \in I}$ is independent with $\mathbb{E}[X_{n,l}] = 0$ and

$$\sigma_{n,k}^2 \triangleq \mathbb{E}|X_{n,l}|^2 = \sum_{i=1}^{d}\sum_{j=1}^{d} \theta_i\theta_j G_{\{p^{-n}t_i\},l} G_{\{p^{-n}t_j\},l}.$$

Furthermore, it can be seen that for large $n \in \mathbb{N}$

$$s_n^2 \triangleq \sum_{l \in I} \sigma_{n,k}^2 = \sum_{i=1}^{d}\sum_{j=1}^{d} \theta_i\theta_j \sum_{l \in I} G_{\{p^{-n}t_i\},l} G_{\{p^{-n}t_j\},l} = p^n \sum_{i=1}^{d}\sum_{j=1}^{d} \theta_i\theta_j B(t_i, t_j) - \frac{1}{p}$$

$$\Gamma_n \triangleq \sum_{l \in I} \mathbb{E}|X_{n,l}|^3 \le C(n+1)^3 p^n \mathbb{E}|Z_0|^3 \Big(\sum_{j=1}^{d} |\theta_j|\Big)^3,$$

where C is a constant depending on p, $t_1, \ldots, t_d$; the inequality comes from a tedious but straightforward computation once we observe that for large $n \in \mathbb{N}$ with $|t_1|_p > p^{-n}$ we have $|\{p^{-n}t_i\}|_p = p^n|t_i|_p$ and $G_{\{p^{-n}\}t_i,l} = 0 \ \forall |l|_p > p^n|t_i|_p$.

We now have $\Gamma_n/s_n^3 \to 0$ $(n \to \infty)$ and it follow from Liapounov's central limit theorem that

$$\frac{1}{s_n}\sum_{l\in I} X_{n,l} \to N(0,1)$$

and hence

$$\sum_{j=1}^{d} \theta_j W^{(n)}(t_j) = p^{-n/2}\sum_{l\in I} X_{n,l} \to N(0, \theta'\Sigma(t_1,\ldots,t_d)\theta),$$

which proves $\varphi^{(n)}(\theta) \to \exp(-\theta'\Sigma(t_1,\ldots,t_d)\theta/2)$. □

Open Problem. Prove that $\{\mu^{(n)}\}_{n\in\mathbb{N}}$ is tight.

The key ingredient of the tightness of $\{\mu^{(n)}\}_{n\in\mathbb{N}}$ for classical random walk is the Kolmogorov-Doob maximal inequality, which essentially improves Čebyšev's inequality. The heart of the proof of the maximal inequality, in turn, is the notion of first hitting time. As far as I know, there exists no p-adic analogue of first hitting time.