$p$-adic Brownian Motion over $\mathbb{Q}_p$ *

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Earlier Research on $p$-adic Brownian Motion


- K. Kamizono, *Symmetric Stochastic Integrals with respect to $p$-adic Brownian Motion*, To Appear in Stochastics.
Plan of This Talk

• Present an explicit construction of $p$-adic Brownian motion over $\mathbb{Q}_p$ using the Paley-Wiener method.

• Introduce $p$-adic random walk over $\mathbb{Q}_p/\mathbb{Z}_p$.

• Discuss the weak convergence of $p$-adic random walk over to $p$-adic Brownian motion (unfinished).

Key words: Brownian motion, $p$-adic analysis, white noise theory

AMS 2000 subject classification: Primary 60G15, 60H40; secondary 60G20, 46S10.
Notation

Write

\[ I_n \triangleq \{ t = t_n p^{-n} + \ldots + t_j p^{-1} \mid t_j \in \{0, 1, \ldots, p-1\} \}, \quad I \triangleq \bigcup_{n \in \mathbb{N}} I_n \cong \mathbb{Q}_p / \mathbb{Z}_p. \]

Every \( t \in \mathbb{Q}_p \) can be uniquely decomposed as

\[ t = [t] + \{t\}, \quad [t] \in \mathbb{Z}_p : integer \text{ part}, \quad \{t\} \in I : fractional \text{ part}. \]

Denote by \( \chi \) the normalized additive character of \( \mathbb{Q}_p \):

\[ \chi(t) \triangleq e^{2\pi \sqrt{-1} \{t\}}, \quad t \in \mathbb{Q}_p. \]

Let \( L^2(\mathbb{Q}_p) \) be the space of square-integrable, complex-valued functions over \( \mathbb{Q}_p \) with respect to the normalized Haar measure. Let \( \mathscr{D}(\mathbb{Q}_p) \) be the space of Schwartz-Bruhat
functions over $\mathbb{Q}_p$ and let $\mathcal{D}'(\mathbb{Q}_p)$ be its topological dual. Denote by $\hat{u}$ the Fourier transform of $u$:

$$
\hat{\varphi}(k) \triangleq \int_{\mathbb{Q}_p} \varphi(t) \chi(kt) \, dt, \quad k \in \mathbb{Q}_p, \quad \text{for} \quad \varphi \in \mathcal{D}(\mathbb{Q}_p)
$$

$$
\langle \hat{f}, \varphi \rangle \triangleq \langle f, \hat{\varphi} \rangle, \quad \varphi \in \mathcal{D}(\mathbb{Q}_p), \quad \text{for} \quad f \in \mathcal{D}'(\mathbb{Q}_p).
$$

Denote by $\mathcal{D}$ the first order Vladimirov differential operator:

$$
\mathcal{D}f \triangleq f^{-1} * f, \quad \text{for} \quad f \in \mathcal{D}'(\mathbb{Q}_p),
$$

if the right-hand-side exists, where $*$ is the convolution operator and

$$
\langle f^{-1}, \varphi \rangle \triangleq -\frac{p^2}{p+1} \left[ \int_{\mathbb{Z}_p} \frac{\varphi(t) - \varphi(0)}{|t|^2_p} \, dt + \int_{\mathbb{Q}_p \setminus \mathbb{Z}_p} \frac{\varphi(t)}{|t|^2_p} \, dt - \frac{1}{p} \varphi(0) \right], \quad \varphi \in \mathcal{D}(\mathbb{Q}_p).
$$

For any subset $A \subseteq \mathbb{Q}_p$, denote by $1_A$ the indicator function of $A$:

$$
1_A(t) \triangleq \begin{cases} 1 & \text{for} \ t \in A \\ 0 & \text{for} \ t \in \mathbb{Q}_p \setminus A \end{cases}
$$
**p-adic White Noise over \( \mathbb{Q}_p \)**

Fix a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). All random variables will be complex-valued and defined on \((\Omega, \mathcal{F}, \mathbb{P})\) unless stated otherwise.

**Definition 1 (p-adic White Noise).** A \( p \)-adic white noise over \( \mathbb{Q}_p \) is a random variable \( X \) taking values in \( D'(\mathbb{Q}_p) \) such that the family \( \{\langle X, \varphi \rangle\}_{\varphi \in \mathcal{D}(\mathbb{Q}_p)} \) is Gaussian with mean 0 and covariance

\[
E[\langle X, \varphi \rangle \overline{\langle X, \psi \rangle}] = \int_{\mathbb{Q}_p} \varphi(t)\overline{\psi(t)}dt \quad \forall \varphi, \psi \in \mathcal{D}(\mathbb{Q}_p).
\]

Introduce the family of functions \( \{\psi_{k,l}\}_{k,l \in I} \subseteq \mathcal{D}(\mathbb{Q}_p) \) given by

\[
\psi_{k,l}(t) \triangleq 1_{k+\mathbb{Z}_p}(t)\chi(l(t-k)), \quad t \in \mathbb{Q}_p.
\]

**Lemma 2.** The family \( \{\psi_{k,l}\}_{k,l \in I} \) forms a complete orthonormal system of \( L^2(\mathbb{Q}_p) \).
Proof. It is easy to check that \( \{ \psi_{k,l} \}_{k,l \in I} \) forms an orthonormal system of \( L^2(Q_p) \). Let us prove completeness. Let \( \varphi \in \mathcal{D}(Q_p) \). Take a large \( m \in \mathbb{N} \) such that \( \text{supp}(\varphi) \cup \text{supp}(\hat{\varphi}) \subseteq p^{-m} \mathbb{Z}_p \). From the Parseval-Steklov equation, we have

\[
\| \varphi \|^2_{L^2(Q_p)} = \| \hat{\varphi} \|^2_{L^2(Q_p)} = \sum_{l \in I_m} \int_{l+\mathbb{Z}_p} \left| \sum_{k \in I_m} \int_{k+\mathbb{Z}_p} \varphi(t) \chi(ut) dt \right|^2 \, du.
\]

Now, if \( u \in l + \mathbb{Z}_p \) and \( t \in k + \mathbb{Z}_p \), we have \( \chi((u - l)(t - k)) = 1 \). Thus, for \( u \in l + \mathbb{Z}_p \),

\[
\left| \sum_{k \in I_m} \int_{k+\mathbb{Z}_p} \varphi(t) \chi(ut) dt \right|^2 = \left| \sum_{k \in I_m} \chi(uk) \int_{k+\mathbb{Z}_p} \varphi(t) \chi(l(t - k)) dt \right|^2 = \sum_{k_1, k_2 \in I_m} \chi(u(k_1 - k_2))(\varphi, \psi_{k_1, -l})_{L^2(Q_p)}(\varphi, \psi_{k_2, -l})_{L^2(Q_p)}.
\]
It follows from (3) and (4) that

\[
\|\varphi\|_{L^2(Q_p)}^2 = \sum_{k_1,k_2,l \in I_m} (\varphi, \psi_{k_1,-l})_{L^2(Q_p)} (\varphi, \psi_{k_2,-l})_{L^2(Q_p)} \int_{l+\mathbb{Z}_p} \chi(u(k_1 - k_2)) du
= \sum_{k,l \in I_m} (\varphi, \psi_k,-l)_{L^2(Q_p)} (\varphi, \psi_k,-l)_{L^2(Q_p)}.
\]

Finally, let \( m \to \infty \) to obtain the result. \( \square \)

**Proposition 3.** The probability distribution of \( p \)-adic white noise \( X \) is not supported on the space \( \mathcal{E}'(Q_p) \) of Schwartz distributions with compact support.

**Proof.** Suppose that \( X \in \mathcal{E}'(Q_p) \) a.s. Then, for a.e. \( \omega \in \Omega \), there exists \( n_\omega \in \mathbb{N} \) such that \( \text{supp } X(\omega) \subseteq p^{-n_\omega} \mathbb{Z}_p \). Now, since \( \text{supp } \psi_{k,l} \subseteq k + \mathbb{Z}_p \), we have \( \langle X(\omega), \psi_{k,l} \rangle = 0 \) for \( k \in I \setminus I_{n_\omega} \). This implies that \( \langle X, \psi_{k,l} \rangle \to 0 \) as \( |k|_p \to \infty \) a.s., and therefore,

\[
e^{-\theta^2/4} = \mathbb{E} \left[ e^{\sqrt{-1} \theta \text{ Re } \langle X, \psi_{k,l} \rangle} \right] \to 1 \quad (|k|_p \to \infty) \quad \forall \theta \in \mathbb{R},
\]

a contradiction. \( \square \)
**$p$-adic Brownian Motion over $\mathbb{Q}_p$**

**Definition 4** ($p$-adic Brownian Motion over $\mathbb{Q}_p$). A $p$-adic Brownian motion over $\mathbb{Q}_p$ is a stochastic process $\{W(t)\}_{t \in \mathbb{Q}_p}$ such that

(i) the family $\{W(t)\}_{t \in \mathbb{Q}_p}$ is Gaussian with mean 0 and covariance

\[
E[W(t)W(s)] = \frac{p + 1}{p^2}(|t|_p + |s|_p - |t - s|_p) \quad \forall t, s \in \mathbb{Q}_p;
\]

(ii) the sample path $t \mapsto W(t, \omega)$ is continuous for every fixed $\omega \in \Omega$.

**Remark 5.** Similar properties to the classical Brownian motion with real time index:

(i) $E[W(t)W(s)] = \frac{p + 1}{p^2} \min(|t|_p, |s|_p)$ if $|t|_p \neq |s|_p$.

(ii) $W(t_2) - W(t_1)$ and $W(t_4) - W(t_3)$ are mutually independent if $0 \leq |t_1|_p \leq |t_2|_p < |t_3|_p \leq |t_4|_p$. 
Construction of $p$-adic Brownian Motion over $\mathbb{Q}_p$

Consider the stochastic equation:

\begin{equation}
D\Phi = X(\omega), \quad \Phi(0) = 0,
\end{equation}

where $X$ is $p$-adic white noise over $\mathbb{Q}_p$. If $X(\omega) \in \mathcal{E}'(\mathbb{Q}_p)$, then (6) would have a unique solution. However, as we have seen, $\mathbb{P}(\omega : X(\omega) \in \mathcal{E}'(\mathbb{Q}_p)) \neq 1$. Thus, (6) does not have a solution in almost sure sense. We will take the Paley-Wiener method to construct $p$-adic Brownian Motion over $\mathbb{Q}_p$. This is a direct generalization of [A. Kh. Bikulov and I.V. Volovich, Izv. Math. 61 (1997) pp. 537–552].

Suppose first that there exists on $(\Omega, \mathcal{F}, \mathbb{P})$ an independent family $\{Z_{k,l}\}_{k,l \in I}$ of standard Gaussian random variables. For each $\varphi \in \mathcal{D}(\mathbb{Q}_p)$, let

\begin{equation}
\langle X(\omega), \varphi \rangle \triangleq \sum_{k,l \in I} Z_{k,l}(\omega) \langle \psi_{k,-l}, \varphi \rangle \quad \omega \in \Omega.
\end{equation}

The family $\{X\varphi\}_{\varphi \in \mathcal{D}(\mathbb{Q}_p)}$ is clearly $p$-adic white noise.
Next, consider the following function

\( G(t, s) \triangleq -\frac{1 - p^{-1}}{\log p} \left( \log |t - s|_p - \log |s|_p \right), \quad (t, s) \in \mathbb{Q}_p \times \mathbb{Q}_p, \; t \neq s, \; s \neq 0. \)

For every fixed \( t \in \mathbb{Q}_p, \) the function \( G(t, \cdot) \) belongs to \( \mathbb{L}^1(\mathbb{Q}_p) \) with Fourier transform

\( \hat{G}(t, \cdot)(k) = \frac{\chi(kt) - 1}{|k|_p} \quad \forall k \in \mathbb{Q}_p \setminus \{0\} \) \quad and \quad \hat{G}(t, \cdot)(0) = 0. \)

Let us consider the integral operator

\( T \varphi(t) \triangleq \int_{\mathbb{Q}_p} G(t, s) \varphi(s) ds, \quad t \in \mathbb{Q}_p. \)

For every \( f \in \mathcal{D}(\mathbb{Q}_p), \) \( u = Tf \) is the unique solution to

\( D u = f \quad \text{on} \; \mathbb{Q}_p, \quad u(0) = 0. \)
We would like to solve, in some sense, the stochastic equation:

\[ \mathbf{D}W = X, \quad W(0) = 0, \]

where \( X \) is \( p \)-adic white noise:

\[ X = \sum_{k,l \in I} Z_{k,l} \psi_{k,-l}. \]

If \( X \) were random variable taking values in \( \mathcal{D}(\mathbb{Q}_p) \), the solution could be written as

\[ W = T_X = \sum_{k,l \in I} Z_{k,l} T \psi_{k,-l}. \]

With this observation in mind, it is natural to take the limit

(12) \[ W(t) \equiv \lim_{n \to \infty} \sum_{k,l \in I_n} Z_{k,l} T \psi_{k,-l}(t), \quad t \in \mathbb{Q}_p \]

as a candidate of \( p \)-adic Brownian motion over \( \mathbb{Q}_p \).
Theorem 6 (Existence of $p$-adic Brownian Motion over $\mathbb{Q}_p$). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which there exists an independent family $\{Z_{k,l}\}_{k,l \in I}$ of standard Gaussian random variables. Let $\{\psi_{k,l}\}_{k,l \in I}$ be the complete orthonormal system of $L^2(\mathbb{Q}_p)$ given by (2), and let $T$ be the integral operator given by (10) with integral kernel (8). Then,

(i) the limit in (12) exists in $L^2(\Omega)$ for every fixed $t \in \mathbb{Q}_p$;

(ii) the family $\{W(t)\}_{t \in \mathbb{Q}_p}$ is Gaussian with mean 0 and covariance

$$
\mathbb{E}[W(t)W(s)] = \frac{p + 1}{p^2}(|t|_p + |s|_p - |t - s|_p).
$$

(iii) there exists a continuous modification of $\{W(t)\}_{t \in \mathbb{Q}_p}$.

Proof. First, let us calculate $T\psi_{k,-l}$. For $l = 0$, we have $\psi_{k,0}(s) = 1_{k+z_p}(s)$ and thus, $T\psi_{k,0}(t) = G(t,s) \cdot 1_{z_p}(k)$. Let $l \neq 0$. In this case,

$$
T\psi_{k,-l}(t) = \chi(lk) \int_{\mathbb{Q}_p} 1_{k+z_p}(s) G(t,s) \chi(-ls) ds
$$

$$
= \chi(lk)(1_{k+z_p} \cdot G(t,\cdot))^(-l) = \chi(lk)(\hat{1}_{k+z_p} * \hat{G}(t,\cdot))(-l).
$$
Now, since $\hat{1}_{k+Z_p} = \int_{k+Z_p} \chi(us)ds = \chi(uk)1_{Z_p}(u)$, it follows from (9) that

$$\hat{1}_{k+Z_p} \ast \hat{G}(t, \cdot)(-l) = \int_{Q_p} 1_{Z_p}(l+u)\chi(-(l+u)k)\frac{\chi(ut) - 1}{|u|_p}du$$

$$= \int_{Z_p} \chi(-vk)\frac{\chi((v-l)t) - 1}{|v-l|_p}dv$$

$$= \begin{cases} 
\frac{\chi(-lt)1_{Z_p}(t) - 1}{|l|_p} & \text{if } k = 0 \\
\chi(-lt)1_{k+Z_p}(t) & \text{if } k \neq 0 
\end{cases} .$$

Combining (13) and (14), we obtain

$$T_{\psi_{k,-l}}(t) = \begin{cases} 
\frac{\chi(-l[t])1_{Z_p}(t) - 1}{|l|_p} & \text{if } k = 0 \\
\chi(-l[t])1_{k+Z_p}(t) & \text{if } k \neq 0 
\end{cases} .$$
The series (12) then becomes

\begin{equation}
W(t) = \lim_{n \to \infty} \left[ \sum_{k \in I_n} Z_{k,0}(G(t, \cdot) * 1_{Z_p})(k) + \sum_{l \in I_n \setminus \{0\}} \frac{Z_{\{t\} \cup \chi(-l[t]) - Z_{0,l}}}{|l|_p} \right].
\end{equation}

To prove the convergence in $L^2(\Omega)$ of the right-hand-side of (16), notice first that the first term of (16) stabilizes for large $n$ because the summand vanishes if $|k|_p > |t|_p$.

To see the convergence of the second term, we calculate as

\[
\mathbb{E} \left| \sum_{l \in I_n \setminus \{0\}} \frac{Z_{\{t\} \cup \chi(-l[t]) - Z_{0,l}}}{|l|_p} \right|^2 = \begin{cases} 
\sum_{l \in I_n \setminus \{0\}} \frac{(\chi(-lt) - 1)(\chi(lt) - 1)}{|l|^2_p} \to \frac{2(p + 1)}{p^2} |t|_p & (n \to \infty) \quad \text{if } t \in \mathbb{Z}_p \\
\sum_{l \in I_n \setminus \{0\}} \frac{\chi(-l[t])\chi(l[t])}{|l|^2_p} + \sum_{l \in I_n \setminus \{0\}} \frac{1}{|l|^2_p} \to \frac{2}{p} & (n \to \infty) \quad \text{if } t \in \mathbb{Q}_p \setminus \mathbb{Z}_p
\end{cases}
\]

This proves (i).
It is obvious that \( \{W(t)\}_{t \in \mathbb{Q}_p} \) is Gaussian with mean 0. Let us calculate the covariance. If \( t, s \in \mathbb{Z}_p \), the right-hand-side of (16) coincides, except for the constant multiple, with the \( p \)-adic Brownian motion over \( \mathbb{Z}_p \) of Bikulov and Volovich (1997). Thus, (5) holds for \( t, s \in \mathbb{Z}_p \). If \( t \in \mathbb{Z}_p \) and \( s \in \mathbb{Q}_p \setminus \mathbb{Z}_p \), then

\[
\mathbb{E}[W(t)W(s)] = -\lim_{l \in I_n \setminus \{0\}} \frac{\chi(lt) - 1}{|l|^2_p} = \frac{p + 1}{p^2} |t|_p,
\]

and the right-hand-side coincides with that of (5). If \( t, s \in \mathbb{Q}_p \setminus \mathbb{Z}_p \), write

(17) \[
\mathbb{E}[W(t)W(s)] = S_1(t, s) + S_2(t, s),
\]

where \( S_1(t, s) \) and \( S_2(t, s) \) are given by

\[
S_1(t, s) \triangleq \begin{cases} 
\sum_{l \in I \setminus \{0\}} \frac{\chi(-l([t] - [s]))) + 1}{|l|^2_p} = \frac{2}{p} - \frac{p + 1}{p^2} |t - s|_p & \text{if } |t - s|_p \leq 1 \\
\sum_{l \in I \setminus \{0\}} \frac{1}{|l|^2_p} = \frac{1}{p} & \text{if } |t - s|_p > 1
\end{cases}
\]
as well as

\[ S_2(t, s) \overset{\Delta}{=} \sum_{k \in I} (G(t, \cdot) * 1_{\mathbb{Z}_p})(k)(G(s, \cdot) * 1_{\mathbb{Z}_p})(k) \]

\[ = \int_{\mathbb{Q}_p} (G(t, \cdot) * 1_{\mathbb{Z}_p})(k)(G(s, \cdot) * 1_{\mathbb{Z}_p})(k) \, dk \]

\[ = \int_{\mathbb{Q}_p} \left( \int_{\mathbb{Z}_p} G(t, k - u) \, du \right) \left( \int_{\mathbb{Z}_p} G(s, k - v) \, dv \right) \, dk \]

\[ = \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} \left( \int_{\mathbb{Q}_p} G(t, k - u)G(s, k - v) \, dk \right) \, dudv \]

\[ = \begin{cases} \frac{p + 1}{p^2}(|t|_p + |s|_p) - \frac{2}{p} & \text{if } |t - s|_p \leq 1 \\ \frac{p + 1}{p^2}(|t|_p + |s|_p - |t - s|_p) - \frac{1}{p} & \text{if } |t - s|_p > 1 \end{cases} \]

It follows that (5) is valid for \( t, s \in \mathbb{Q}_p \setminus \mathbb{Z}_p \) as well. This proves \((ii)\).
Finally, for each $k \in I$, let

$$W_k(t) \triangleq W(t + k), \quad t \in \mathbb{Z}_p.$$ 

Then, we see

$$\mathbb{E} |W_k(t) - W_k(s)|^{2n} = \left(\frac{4(p + 1)^4}{p^8}\right)^n n! |t - s|^n_p \quad \forall t, s \in \mathbb{Z}_p, \forall n \in \mathbb{N},$$

and therefore it follows from Lemma 7 below that there exists for every $k \in I$ a continuous modification $\{\tilde{W}_k(t)\}_{t \in \mathbb{Z}_p}$ for $\{W_k(t)\}_{t \in \mathbb{Z}_p}$. Since the $I$ is countable and $\mathbb{Q}_p$ is the union of disjoint open subsets $k + \mathbb{Z}_p$ with $k \in I$, we conclude that $\{\tilde{W}_{\{t\}([t])}\}_{t \in \mathbb{Q}_p}$ is a continuous modification of $\{W(t)\}_{t \in \mathbb{Q}_p}$ \hfill $\square$
Lemma 7 (The Kolmogorov-Čentsov Criterion). Let \( \{X(t)\}_{t \in \mathbb{Z}_p} \) be a stochastic process for which there exist constants \( \alpha > 0, \beta > 0 \) and \( C > 0 \) such that

\[
\mathbb{E} |X(t) - X(s)|^\alpha \leq C |t - s|_p^{1+\beta} \quad \forall t, s \in \mathbb{Z}_p.
\]

Then, \( \{X(t)\}_{t \in \mathbb{Z}_p} \) has a continuous modification \( \{\tilde{X}(t)\}_{t \in \mathbb{Z}_p} \) which is locally Hölder continuous of exponent \( \gamma \) for every \( 0 < \gamma < \beta/\alpha \), that is, there exist a constant \( \delta > 0 \) and a positive random variable \( h \) such that

\[
\sup_{0 < |t - s|_p < h(\omega)} \frac{|\tilde{X}(t, \omega) - \tilde{X}(s, \omega)|}{|t - s|_p^\gamma} \leq \delta, \quad \forall \omega \in \Omega.
\]

Proof. See Appendix of [K. Kamizono “Symmetric Stochastic Integrals with respect to \( p \)-adic Brownian Motion”, To Appear in Stochastics]. \( \square \)
$p$-adic Difference Equations over $I \simeq \mathbb{Q}_p/\mathbb{Z}_p$

Let us consider a discrete time analogue of the Vladimirov differential operator. Consider the spaces

$$\mathcal{D}' \triangleq \left\{ \varphi = \{ \varphi_k \}_{k \in I} \big| \varphi_k \in \mathbb{C} \quad \forall k \in I \right\}$$

$$\mathcal{D} \triangleq \left\{ \varphi \in \mathcal{D}' \bigg| \text{supp } \varphi \subseteq I_n \quad \text{for some } n \in \mathbb{N} \right\}.$$

We say that $\varphi \in \mathcal{D}'$ is differenceable if the limit

$$\Delta \varphi_k \triangleq -\frac{p^2}{p + 1} \lim_{n \to \infty} \sum_{l \in I_n \setminus \{0\}} \varphi_{k+l} \frac{|l|_p^2}{p} + \frac{p}{p + 1} \varphi_k$$

exists for all $k \in I$. In this case, we call $\Delta \varphi \triangleq \{ \Delta \varphi_k \}_{k \in I}$ the difference of $\varphi$. 

\[(20)\]
Let

\[
G_{k,l} \triangleq \begin{cases}
-\frac{1-p^{-1}}{\log p} \left( \log |k-l|_p - \log |l|_p \right) & \text{if } k \neq l, \, l \neq 0 \\
-\frac{1-p^{-1}}{\log p} \left( \log |k|_p + \frac{p^{-1}}{1-p^{-1}} \log p \right) & \text{if } k \neq l, \, l = 0 \\
-\frac{1-p^{-1}}{\log p} \left( -\frac{p^{-1}}{1-p^{-1}} - \log |k|_p \right) & \text{if } k = l, \, l \neq 0 \\
0 & \text{if } k = l = 0
\end{cases}
\]

(21)

Notice that $G_{k,.} \in \mathcal{D}$ for each $k \in I$ and

\[
G_{\varphi_k} \triangleq \sum_{l \in I} G_{k,l} \varphi_l, \quad k \in I
\]

is well-defined for $\varphi \in \mathcal{D}'$.

**Theorem 8.** If $\varphi \in \mathcal{D}$ and $c \in \mathbb{C}$, then there exists a unique solution to

\[
\Delta X = \varphi, \quad X_0 = c
\]

(22)

and the solution is written as

\[
X = G\varphi + c.
\]

(23)
\textbf{p-adic Random Walk over } I \simeq \mathbb{Q}_p/\mathbb{Z}_p

\textbf{Definition 9.} A \textit{p-adic symmetric random walk} is the family \( S = \{S_k\}_{k \in I} \) of random variable given by

\begin{equation}
S_k(\omega) \triangleq GZ_k(\omega) = \sum_{l \in I} G_{k,l} Z_l(\omega), \quad k \in I, \ \omega \in \Omega,
\end{equation}

where \( Z = \{Z_k\}_{k \in I} \) is an independent family of random variables with \( \mathbb{E}[Z_k] = 0, \ \mathbb{E}|Z_k|^2 = 1 \) and \( \mathbb{E}|Z_k|^3 < \infty \) on some probability space \((\Omega, \mathcal{F}, \mathbb{P})\).

It can be shown that

(i) \( S \) is differenceable a.s. with \( \Delta S = Z \), and

(ii) \( S \) is a centered Gaussian random variable with

\begin{align*}
\mathbb{E}[S_k S_l] &= \frac{p + 1}{p^2}(|k|_p + |l|_p - |k - l|_p) - \frac{1}{p}, \quad \text{if } k \neq l \\
\mathbb{E}|S_k|^2 &= \frac{2(p + 1)}{p^2} |k|_p - \frac{2}{p}.
\end{align*}
Weak Convergence

Let $S = \{S_k\}_{k \in I}$ be a $p$-adic random walk. For each $n \in \mathbb{N}$, let

$$W^{(n)}(t) \triangleq p^{-n/2}S_{\{p^{-n}t\}}, \quad t \in \mathbb{Q}_p.$$  \hspace{1cm} (26)

Then, $\{W^{(n)}(t)\}_{t \in \mathbb{Q}_p}$ induces a probability measure $\mu^{(n)}$ on the space $(C'(\mathbb{Q}_p), \mathcal{B}(C'(\mathbb{Q}_p)))$ of continuous functions.

**Conjecture.** As $n \to \infty$, the sequence $\mu^{(n)}$ converges weakly to a probability measure $\mu$ with respect to which the process $\{w(t)\}_{t \in \mathbb{Q}_p}$ given by $C'(\mathbb{Q}_p) \ni w \mapsto w(t)$ is a $p$-adic Brownian motion.

We need to show:

(i) Every finite dimensional distribution of $\{\mu^{(n)}\}_{n \in \mathbb{N}}$ converges weakly to that of $\mu$.

(ii) $\{\mu^{(n)}\}_{n \in \mathbb{N}}$ is tight.

I managed to prove (i), but (ii) is still open.
Lemma 10. For every distinct $t_1, \ldots, t_d \in \mathbb{Q}_p$, the probability distribution of $(W^{(n)}(t_1), \ldots, W^{(n)}(t_d))$ converges weakly to the centered Gaussian with covariance matrix $\Sigma(t_1, \ldots, t_d) \triangleq (B(t_i, t_j))$ with

\begin{equation}
B(t, s) \triangleq \frac{p + 1}{p^2}(|t|_p + |s|_p - |t - s|_p), \quad t, s \in \mathbb{Q}_p.
\end{equation}

Proof. For simplicity, we consider the real-valued case. It suffices to show $\varphi^{(n)}(\theta) \to \exp(-\theta' \Sigma(t_1, \ldots, t_d) \theta/2)$ $(n \to \infty)$ $\forall \theta \in \mathbb{R}^d$, where $\varphi^{(n)}(\theta) \triangleq \mathbb{E}[\exp(\sqrt{-1} \sum_{j=1}^d \theta_j W^{(n)}(t_j))]$. Without loss of generality, we may assume $|t_1|_p \leq \cdots \leq |t_d|_p$. By definition,

$$
\sum_{j=1}^d \theta_j W^{(n)}(t_j) = p^{-n/2} \sum_{j=1}^d \theta_j \sum_{l \in I} G_{\{p^{-n}t_j\}, l} Z_l = p^{-n/2} \sum_{l \in I} X_{n, l},
$$

where

$$
X_{n, l} \triangleq \sum_{j=1}^d \theta_j G_{\{p^{-n}t_j\}, l} Z_l.
$$
For each $n \in \mathbb{N}$, the family $\{X_{n,l}\}_{l \in I}$ is independent with $\mathbb{E}[X_{n,l}] = 0$ and

$$\sigma_{n,k}^2 \triangleq \mathbb{E}|X_{n,l}|^2 = \sum_{i=1}^{d} \sum_{j=1}^{d} \theta_i \theta_j G_{\{p^{-n}t_i\},l} G_{\{p^{-n}t_j\},l}.$$ 

Furthermore, it can be seen that for large $n \in \mathbb{N}$

$$s_n^2 \triangleq \sum_{l \in I} \sigma_{n,k}^2 = \sum_{i=1}^{d} \sum_{j=1}^{d} \theta_i \theta_j \sum_{l \in I} G_{\{p^{-n}t_i\},l} G_{\{p^{-n}t_j\},l} = p^n \sum_{i=1}^{d} \sum_{j=1}^{d} \theta_i \theta_j B(t_i, t_j) - \frac{1}{p}$$

$$\Gamma_n \triangleq \sum_{l \in I} \mathbb{E}|X_{n,l}|^3 \leq C(n + 1)^3 p^n \mathbb{E}|Z_0|^3 \left(\sum_{j=1}^{d} |\theta_j|\right)^3,$$

where $C$ is a constant depending on $p$, $t_1, \ldots, t_d$; the inequality comes from a tedious but straightforward computation once we observe that for large $n \in \mathbb{N}$ with $|t_1|_p > p^{-n}$ we have $|\{p^{-n}t_i\}|_p = p^n |t_i|_p$ and $G_{\{p^{-n}t_i\},l} = 0 \ \forall |l|_p > p^n |t_i|_p$.

We now have $\Gamma_n / s_n^3 \to 0$ ($n \to \infty$) and it follow from Liapounov’s central limit
theorem that
\[ \frac{1}{s_n} \sum_{l \in I} X_{n,l} \rightarrow N(0, 1) \]
and hence
\[ \sum_{j=1}^{d} \theta_j W^{(n)}(t_j) = p^{-n/2} \sum_{l \in I} X_{n,l} \rightarrow N(0, \theta' \Sigma(t_1, \ldots, t_d) \theta), \]
which proves \( \varphi^{(n)}(\theta) \rightarrow \exp(-\theta' \Sigma(t_1, \ldots, t_d) \theta/2) \).

Open Problem. Prove that \( \{\mu^{(n)}\}_{n \in \mathbb{N}} \) is tight.

The key ingredient of the tightness of \( \{\mu^{(n)}\}_{n \in \mathbb{N}} \) for classical random walk is the Kolmogorov-Doob maximal inequality, which essentially improves Čebyšev’s inequality. The heart of the proof of the maximal inequality, in turn, is the notion of first hitting time. As far as I know, there exists no \( p \)-adic analogue of first hitting time.